

The ring structure for equivariant twisted K -theory

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Abstract

We prove, under some mild conditions, that the equivariant twisted K -theory group of a crossed module admits a ring structure if the twisting 2-cocycle is 2-multiplicative. We also give an explicit construction of the transgression map $T_1 : H^*(\Gamma_\bullet; \mathcal{A}) \rightarrow H^{*-1}((N \rtimes \Gamma)_\bullet; \mathcal{A})$ for any crossed module $N \rightarrow \Gamma$ and prove that any element in the image is ∞ -multiplicative. As a consequence, we prove that, under some mild conditions, for a crossed module $N \rightarrow \Gamma$ and any $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$, that the equivariant twisted K -theory group $K_{e,\Gamma}^*(N)$ admits a ring structure. As an application, we prove that for a compact, connected and simply connected Lie group G , the equivariant twisted K -theory group $K_{[c],G}^*(G)$, defined as the K -theory group of a certain groupoid C^* -algebra, is endowed with a canonical ring structure $K_{[c],G}^{i+d}(G) \otimes K_{[c],G}^{j+d}(G) \rightarrow K_{[c],G}^{i+j+d}(G)$, where $d = \dim G$ and $[c] \in H^2((G \rtimes G)_\bullet; \mathcal{S}^1)$. The relation with Freed-Hopkins-Teleman theorem [26] still needs to be explored.

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1 Introduction

A great deal of interest in twisted equivariant K -theory has emerged due to its close connection to string theory [46, 47]. In particular, the recent work of Freed–Hopkins–Teleman [24, 25, 26, 27, 28] concerning the relationship between the twisted equivariant K -theory of compact Lie groups and Verlinde algebras has inspired a great deal of activities in this subject. It now becomes increasingly important to develop a general framework which allows one to study the ring structure of twisted equivariant K -theory groups and in particular to investigate the general criteria which guarantee the existence of such a ring structure.

This paper serves this purpose. More precisely, in this paper we examine the conditions under which the twisted K -theory groups of a crossed module admit a ring structure. Recall that a crossed module is a groupoid morphism

$$\begin{array}{ccc} N_1 & \xrightarrow{\varphi} & \Gamma_1 \\ \Downarrow & & \Downarrow \\ N_0 & \xrightarrow{=} & \Gamma_0 \end{array}$$

where $N_1 \rightrightarrows N_0$ is a bundle of groups, together with an action of Γ on N by automorphisms satisfying some compatibility conditions (see Definition 2.2). A standard example of a crossed module is as follows. Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a groupoid and $S\Gamma_1 = \{g \in \Gamma_1 \mid s(g) = t(g)\}$ be the space of closed loops in Γ_1 . Then the canonical inclusion $S\Gamma \rightarrow \Gamma$, together with the conjugation action of Γ on $S\Gamma$, forms a crossed module. In particular, when Γ is just a Lie group G , $S\Gamma$ is isomorphic to G with the action being by conjugation. In other words, $G \xrightarrow{id} G$ with the conjugation action is a crossed module. Given a crossed module $N \rightarrow \Gamma$, since Γ acts on N , one forms the transformation groupoid (also called the crossed product groupoid) $N \rtimes \Gamma$. In the case that the crossed module is $S\Gamma \rightarrow \Gamma$, the transformation groupoid obtained is called the inertia groupoid and is denoted by $\Lambda\Gamma$. When Γ is a Lie group G , the inertia groupoid is the standard transformation groupoid $G \rtimes G \rightrightarrows G$ with G acting on G by conjugation.

In [43], we developed a general theory of twisted K -theory for differential stacks (see also [2, 3] for the case of quotient stacks). For a Lie groupoid $X_1 \rightrightarrows X_0$ and $\alpha \in H^2(X_\bullet, \mathcal{S}^1)$, the twisted K -theory groups $K_\alpha^*(X)$ are defined to be the K -theory

groups of a certain C^* -algebra $C_r^*(X, \alpha)$ associated to the element α (or an S^1 -gerbe) using groupoid S^1 -central extensions. However, the construction is not canonical and depends on a choice of 2-cocycle $c \in \check{Z}^2(X_\bullet; \mathcal{S}^1)$ representing α , though different choices of c give rise to isomorphic K -theory groups. For the convenience of our investigation, in this paper, we will define twisted K -theory groups using a Čech 2-cocycle instead of a cohomology class so that the twisted K -theory groups $K_c^*(X)$ will be canonically defined. For a Lie groupoid Γ acting on a manifold N , and $c \in \check{Z}^2((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ a 2-cocycle of the corresponding transformation groupoid $N \rtimes \Gamma$, the twisted equivariant K -theory groups are then defined to be

$$K_{c,\Gamma}^i(N) = K_c^i(N \rtimes \Gamma).$$

The main question we study in this paper is: For a crossed module $N \rightarrow \Gamma$, under what condition do the twisted equivariant K -theory groups $K_{c,\Gamma}^i(N)$ admit a ring structure?

The answer is that c needs to be 2-multiplicative. Note that since $N \rightarrow \Gamma$ is a crossed module, $(N_\bullet \rtimes \Gamma)_\bullet$ becomes a bi-simplicial space. Therefore there are two simplicial maps $\partial : \check{C}^p((N_q \rtimes \Gamma)_\bullet; \mathcal{S}^1) \rightarrow \check{C}^{p+1}((N_q \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ and $\partial' : \check{C}^p((N_q \rtimes \Gamma)_\bullet; \mathcal{S}^1) \rightarrow \check{C}^p((N_{q+1} \rtimes \Gamma)_\bullet; \mathcal{S}^1)$. A 2-cocycle $c \in \check{Z}^2((N_1 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ (i.e. $\partial c = 0$) is said to be 2-multiplicative if there exist $b \in \check{C}^1((N_2 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ and $a \in \check{C}^0((N_3 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ such that $\partial' c = \partial b$, and $\partial' b = \partial a$. Such a triple (c, b, a) is called a *multiplicator*. The product structure on $K_{c,\Gamma}^*(N)$ depends on the choice of a multiplicator. The main result of the paper can be summarized as the following

Theorem A. *Let $N \xrightarrow{\varphi} \Gamma$ be a crossed module, where $\Gamma_1 \rightrightarrows \Gamma_0$ is a proper Lie groupoid such that $s : N_1 \rightarrow N_0$ is Γ -equivariantly K -oriented. Assume that (c, b, a) is a multiplicator, where $c \in \check{C}^2((N_1 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$, $b \in \check{C}^1((N_2 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$, and $a \in \check{C}^0((N_3 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$. Then there is a canonical associative product*

$$K_{c,\Gamma}^{i+d}(N) \otimes K_{c,\Gamma}^{j+d}(N) \rightarrow K_{c,\Gamma}^{i+j+d}(N),$$

where $d = \dim N_1 - \dim N_0$.

The main idea of our approach is to transform this geometric problem into a problem of C^* -algebras, for which there are many sophisticated K -theoretic techniques. As the first step, we give a canonical construction of an equivariant S^1 -gerbe (or rather S^1 -central extension), which should be of independent interest.

Theorem B. *Suppose that $\Gamma : \Gamma_1 \rightrightarrows \Gamma_0$ is a Lie groupoid acting on a manifold N via $J : N \rightarrow \Gamma_0$. Let \mathcal{U} be a cover of $(N \rtimes \Gamma)_\bullet$. Then any Čech 2-cocycle $c \in \check{Z}^2(\mathcal{U}, \mathcal{S}^1)$ determines a canonical S^1 -central extension of the form $\tilde{H} \rtimes \Gamma \rightarrow H \rtimes \Gamma \rightrightarrows M$, where $\tilde{H} \rightarrow H \rightrightarrows M$ is a Γ -equivariant S^1 -central extension and $H \rightrightarrows M$ is Morita equivalent to $N \rightrightarrows N$, with the Dixmier-Douady class of the extension equal to $[c] \in \check{H}^2((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$.*

The above theorem allows us to establish a canonical Morita equivalence between the C^* -algebra $C_r^*(N \rtimes \Gamma, c)$ and the crossed product algebra $A_c \rtimes_r \Gamma$, where A_c is a Γ - C^* -algebra (i.e. a C^* -algebra with a Γ -action). This enables us to construct the product structure on $K_{c,\Gamma}^*(N)$ with the help of the Gysin map and the external Kasparov product.

For a Γ -equivariantly K -oriented submersion $f : M \rightarrow N$ between proper Γ -manifolds M and N , the Gysin map is a wrong-way functorial map

$$f_! : K_{f^*c,\Gamma}^i(M) \rightarrow K_{c,\Gamma}^{i+d}(N),$$

where $d = \dim N - \dim M$, which satisfies $g_! \circ f_! = (g \circ f)_!$. It is standard that any K -oriented map $f : M \rightarrow N$ yields a Gysin element $f_! \in KK^d(C_0(M), C_0(N))$ [18, 31]. When Γ is a Lie group, an equivariant version was proved by Kasparov–Skandalis [34, §4.3]: Any Γ -equivariantly K -oriented map $f : M \rightarrow N$ determines an element $f_! \in KK_\Gamma^d(C_0(M), C_0(N))$. A similar argument can be adapted to show that the same assertion holds when Γ is a Lie groupoid and KK_Γ^* is Le Gall’s groupoid equivariant KK -theory [36]. As a consequence, our Gysin map can easily be constructed using such a Gysin element. We note that a different approach to the Gysin map to (non-equivariant) twisted K -theory was recently studied by Carey–Wang [15].

The second ingredient of our construction is the external Kasparov product

$$K_{c,\Gamma}^i(N) \otimes K_{c,\Gamma}^j(N) \rightarrow K_{p_1^*c+p_2^*c,\Gamma}^{i+j}(N_2), \quad (1)$$

where Γ is a proper Lie groupoid, and $p_1, p_2 : N_2 \rightarrow N_1$ are the natural projections. This essentially follows from the usual Kasparov product $KK_\Gamma^i(A, B) \otimes KK_\Gamma^j(C, D) \rightarrow KK_\Gamma^{i+j}(A \otimes_{C_0(\Gamma_0)} C, B \otimes_{C_0(\Gamma_0)} D)$, where A, B, C, D are Γ - C^* -algebras. Here again KK_Γ^* stands for the Le Gall’s groupoid version of the equivariant KK -theory of Kasparov [33, 36].

Theorem A indicates that the ring structure on twisted equivariant K -theory groups relies on “multiplicators”. A natural question now is how multiplicators arise. In the first half part of the paper, we discuss an important construction, the so-called transgression maps, which is a powerful tool to produce “multiplicators”. At the level of cohomology, the transgression map for a crossed module $N \rightarrow \Gamma$ is a map

$$T_1 : H^k(\Gamma_\bullet; \mathcal{S}^1) \rightarrow H^{k-1}((N_1 \rtimes \Gamma)_\bullet; \mathcal{S}^1).$$

For instance, when $k = 2$, one obtains a map $T_1 : H^3(\Gamma_\bullet; \mathcal{S}^1) \rightarrow H^2((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$. Any element in the image of T_1 is 2-multiplicative, so it is reasonable to expect that the corresponding twisted K -theory groups admit a ring structure. To prove this assertion, since our twisted K -theory groups are defined in terms of 2-cocycles, we must study the transgression map more carefully at the cochain level. Therefore we put our construction of the transgression map into a more general perspective which we believe to be of independent interest.

First, to make our construction more transparent and intrinsic, we introduce the notion of \mathcal{C} -spaces and their sheaf cohomology, for a category \mathcal{C} . By a \mathcal{C} -space, we mean a contravariant functor from the category \mathcal{C} to the category of topological spaces. One similarly defines \mathcal{C} -manifolds. Here we are mainly interested in \mathcal{C} -spaces in which \mathcal{C} is equipped with an additional generalized simplicial structure. One standard example of a generalized simplicial category is the simplicial category Δ , whose corresponding \mathcal{C} -spaces are simplicial spaces. Indeed the generalized simplicial structure on \mathcal{C} enables us to define sheaf and Čech cohomology of a \mathcal{C} -space just as one does for simplicial spaces [20, 23]. A relevant generalized simplicial category for our purpose here is the so-called Δ_2 -category, which is an extension of the bi-simplicial category, i.e., $\Delta \times \Delta$. Indeed Δ_2 has the same objects as $\Delta \times \Delta$, but contains more morphisms.

Let $M_{\bullet\bullet}$ be a Δ_2 -space. Then for any fixed $k \in \mathbb{N}$, both $M_{k,\bullet} = (M_{k,l})_{l \in \mathbb{N}}$ and $M_{\bullet,k} = (M_{l,k})_{l \in \mathbb{N}}$ are simplicial spaces. Suppose that $\mathcal{A}_0 \xrightarrow{d} \mathcal{A}_1 \xrightarrow{d} \cdots$ is a complex of abelian sheaves over $M_{\bullet\bullet}$. Let $C^*(M_{\bullet\bullet}; \mathcal{A}_\bullet)$ (resp. $C^*(M_{0,\bullet}; \mathcal{A}_\bullet)$) be its associated differential complex on $M_{\bullet\bullet}$ (resp. $M_{0,\bullet}$). We prove the following

Theorem C.

1. For each $k \in \mathbb{N}$, there is a map

$$T_k : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^{*-k}(M_{k,\bullet}; \mathcal{A}_\bullet)$$

(with $T_0 = \text{Id}$) such that

$$T = \sum_{k \geq 0} T_k : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^*(M_{\bullet,\bullet}; \mathcal{A}_\bullet)$$

is a chain map which therefore induces a morphism

$$T : H^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow H^*(M_{\bullet,\bullet}; \mathcal{A}_\bullet)$$

on the level of cohomology.

2. In particular,

$$T_1 : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^{*-1}(M_{1,\bullet}; \mathcal{A}_\bullet)$$

is an (anti-)chain map and thus induces a morphism

$$T_1 : H^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow H^{*-1}(M_{1,\bullet}; \mathcal{A}_\bullet).$$

3. Similarly, given an abelian sheaf \mathcal{A} over $M_{\bullet,\bullet}$, there is a map

$$T_k : \check{C}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{C}^{*-k}(M_{k,\bullet}; \mathcal{A})$$

(with $T_0 = \text{Id}$) such that

$$T = \sum_{k \geq 0} T_k : \check{C}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{C}^*(M_{\bullet,\bullet}; \mathcal{A})$$

is a chain map which therefore induces a morphism

$$T : \check{H}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{H}^*(M_{\bullet,\bullet}; \mathcal{A}).$$

4. Similarly, for any abelian sheaf \mathcal{A} over $M_{\bullet,\bullet}$,

$$T_1 : \check{C}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{C}^{*-1}(M_{1,\bullet}; \mathcal{A})$$

is an (anti-)chain map and thus induces a morphism

$$T_1 : \check{H}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{H}^{*-1}(M_{1,\bullet}; \mathcal{A}).$$

We call T the total transgression map and T_1 the transgression map.

For a crossed module $N \xrightarrow{\phi} \Gamma$, one shows that $(N \rtimes \Gamma)_{\bullet,\bullet}$ is naturally a Δ_2 -space. In this case, the transgression maps can be described more explicitly.

Theorem D. Let $N \xrightarrow{\phi} \Gamma$ be a crossed module and \mathcal{A}_\bullet a complex of abelian sheaves over $(N \rtimes \Gamma)_{\bullet,\bullet}$.

1. There is a chain map (the total transgression map)

$$T = \sum_k T_k : C^*(\Gamma_\bullet; \mathcal{A}_\bullet) \rightarrow C^*((N \rtimes \Gamma)_\bullet; \mathcal{A}_\bullet).$$

Moreover

$$T_k = \sum_{\sigma \in S_{k,l}} \varepsilon(\sigma) \tilde{f}_\sigma^* : C^{k+l}(\Gamma_\bullet; \mathcal{A}_\bullet) \longrightarrow C^l((N_k \rtimes \Gamma)_\bullet; \mathcal{A}_\bullet),$$

where $S_{k,l}$ denotes the set of (k, l) -shuffles, and the map $\tilde{f}_\sigma : N_k \rtimes \Gamma_l \rightarrow \Gamma_{k+l}$ is given by

$$\tilde{f}_\sigma(x_1, \dots, x_k; g_1, \dots, g_l) = (u_1, \dots, u_{k+l}), \quad (2)$$

where $u_i = g_{\sigma^{-1}(i)}$ if $\sigma^{-1}(i) \geq k+1$, and $u_i = \varphi \left(x_{\sigma^{-1}(i)}^{\prod_{\sigma^{-1}(j) > k, j < i} g_{\sigma^{-1}(j)}} \right)$ otherwise.

2. There is a transgression map

$$T_1 : H^*(\Gamma_\bullet; \mathcal{A}_\bullet) \rightarrow H^{*-1}((N_1 \rtimes \Gamma)_\bullet; \mathcal{A}_\bullet),$$

which is given, on the cochain level, by

$$T_1 = \sum_{i=0}^{p-1} (-1)^i \tilde{f}_i^* : \mathcal{A}^q(\Gamma_p) \rightarrow \mathcal{A}^q(N_1 \rtimes \Gamma_{p-1}).$$

Here the map $\tilde{f}_i : N_1 \rtimes \Gamma_{p-1} \rightarrow \Gamma_p$ is given by

$$\tilde{f}_i(x; g_1, \dots, g_{p-1}) = (g_1, \dots, g_i, \varphi(x)^{g_1 \cdots g_i}, g_{i+1}, \dots, g_{p-1}). \quad (3)$$

Note that the transgression maps have, in various different forms, appeared in the literature before. For instance, for the crossed module $G \xrightarrow{id} G$ with the conjugation action and $\mathcal{A}_\bullet = \Omega^\bullet$, the transgression map $T_1 : H_G^*(\bullet) \rightarrow H_G^{*-1}(G)$ was studied by Jeffrey [32] (see also [37]). The geometric meaning of the transgression $T_1 : \Omega_G^4(\bullet) \rightarrow \Omega_G^3(G)$ was studied by Brylinski–McLaughlin [11]. On the other hand, the suspension map $H_G^4(\bullet, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$, which is the composition of the transgression $T_1 : H_G^4(\bullet, \mathbb{Z}) \rightarrow H_G^3(G, \mathbb{Z})$ with the canonical map $H_G^3(G, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$, was shown by Dijkgraaf–Witten [21] to induce a geometric correspondence between three dimensional Chern-Simons functional and Wess-Zumino-Witten models. Such a correspondence was further explored recently by Carey et. al. [14] using bundle gerbes and was also used in [16] in their study of fusion of D -branes. The transgression map for orbifold cohomology was recently studied by Adem–Ruan–Zhang [1].

Transgression maps T_k can be used to produce multipliers. More precisely, for a crossed module $N \xrightarrow{\varphi} \Gamma$, if $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$ and letting $c = T_1 e \in \check{C}^2((N_1 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$, $b = -T_2 e \in \check{C}^1((N_2 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$, and $a = -T_3 e \in \check{C}^0((N_3 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$, we then prove that (c, b, a) is a multiplier. This fact enables us to construct a canonical ring structure on the K -theory groups twisted by elements in $\check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$. More precisely, for any $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$, $T_1 e \in \check{Z}^2((N_1 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ is 2-multiplicative. Define

$$K_{e, \Gamma}^*(N) := K_{T_1 e, \Gamma}^*(N).$$

Thus we prove

Theorem E. *Let $N \xrightarrow{\varphi} \Gamma$ be a crossed module, where $\Gamma_1 \rightrightarrows \Gamma_0$ is a proper Lie groupoid such that $s : N_1 \rightarrow N_0$ is Γ -equivariantly K -oriented.*

1. *For any $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$, the twisted K -theory group $K_{e,\Gamma}^*(N)$ is endowed with a ring structure*

$$K_{e,\Gamma}^{i+d}(N) \otimes K_{e,\Gamma}^{j+d}(N) \rightarrow K_{e,\Gamma}^{i+j+d}(N),$$

where $d = \dim N_1 - \dim N_0$.

2. *Assume that e and $e' \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$ satisfy $e - e' = \partial u$ for some $u \in \check{C}^2(\Gamma_\bullet; \mathcal{S}^1)$. Then there is a ring isomorphism*

$$\Psi_{e',u,e} : K_{e,\Gamma}^*(N) \rightarrow K_{e',\Gamma}^*(N)$$

such that

- *if $e - e' = \partial u$ and $e' - e'' = \partial u'$, then*

$$\Psi_{e'',u',e' \circ e'} \Psi_{e',u,e} = \Psi_{e'',u+u',e};$$

- *for any $v \in \check{C}^1(\Gamma_\bullet; \mathcal{S}^1)$,*

$$\Psi_{e',u,e} = \Psi_{e',u+\partial v,e}.$$

3. *There is a morphism*

$$H^2(\Gamma_\bullet; \mathcal{S}^1) \rightarrow \text{Aut } K_{e,\Gamma}^*(N).$$

The ring structure on $K_{e,\Gamma}^{+d}(N)$, up to an isomorphism, depends only on the cohomology class $[e] \in H^3(\Gamma_\bullet; \mathcal{S}^1)$. The isomorphism is unique up to an automorphism of $K_{e,\Gamma}^{*+d}(N)$ induced from $H^2(\Gamma_\bullet; \mathcal{S}^1)$.*

As an application, we consider twisted K -theory groups of an inertia groupoid. Let $\Gamma : \Gamma_1 \rightrightarrows \Gamma_0$ be a Lie groupoid and consider the crossed module $S\Gamma \rightarrow \Gamma$. As before, $\Lambda\Gamma : S\Gamma_1 \rtimes \Gamma_1 \rightrightarrows S\Gamma_1$ denotes the inertia groupoid of Γ . Any element in the image of the transgression map $T_1 : H^3(\Gamma_\bullet; \mathcal{S}^1) \rightarrow H^2(\Lambda\Gamma_\bullet; \mathcal{S}^1)$ is 2-multiplicative. Thus one obtains a ring structure on the corresponding twisted K -theory groups. Since $H^3(\Gamma_\bullet; \mathcal{S}^1)$ classifies 2-gerbes, we conclude that the twisted K -theory groups on the inertia stack twisted by a 2-gerbe over the stack admits a ring structure.

Theorem F. *Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a proper Lie groupoid such that $S\Gamma_1$ is a manifold and $S\Gamma_1 \rightarrow \Gamma_0$ is Γ -equivariantly K -oriented (these assumptions hold, for instance, when Γ is proper and étale, or when Γ is a compact, connected and simply connected Lie group). Let $d = \dim S\Gamma_1 - \dim \Gamma_0$.*

1. *For any $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$, the twisted K -theory groups $K_{e,\Gamma}^{*+d}(S\Gamma)$ are endowed with a ring structure*

$$K_{e,\Gamma}^{i+d}(S\Gamma) \otimes K_{e,\Gamma}^{j+d}(S\Gamma) \rightarrow K_{e,\Gamma}^{i+j+d}(S\Gamma).$$

2. *Assume that e and $e' \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$ satisfy $e - e' = \partial u$ for some $u \in \check{C}^2(\Gamma_\bullet; \mathcal{S}^1)$. Then there is a ring isomorphism*

$$\Psi_{e',u,e} : K_{e,\Gamma}^*(S\Gamma) \rightarrow K_{e',\Gamma}^*(S\Gamma)$$

such that

- if $e - e' = \partial u$ and $e' - e'' = \partial u'$, then

$$\Psi_{e'', u', e' \circ} \Psi_{e', u, e} = \Psi_{e'', u + u', e};$$

- for any $v \in \check{C}^1(\Gamma_\bullet; \mathcal{S}^1)$,

$$\Psi_{e', u, e} = \Psi_{e', u + \partial v, e}.$$

3. There is a morphism

$$H^2(\Gamma_\bullet; \mathcal{S}^1) \rightarrow \text{Aut } K_{e, \Gamma}^*(S\Gamma).$$

The ring structure on $K_{e, \Gamma}^{*+d}(S\Gamma)$, up to an isomorphism, depends only on the cohomology class $[c] \in H^3(\Gamma_\bullet; \mathcal{S}^1)$. The isomorphism is unique up to an automorphism of $K_{e, \Gamma}^{*+d}(S\Gamma)$ induced from $H^2(\Gamma_\bullet; \mathcal{S}^1)$.

As a special case, when Γ is a compact, connected and simply connected, simple Lie group G , $SG \cong G$, and the G -action on G is by conjugation, then $T_1 : H^3(G_\bullet, \mathcal{S}^1) \rightarrow H^2((G \rtimes G)_\bullet, \mathcal{S}^1)$ is an isomorphism and $H^2(G_\bullet, \mathcal{S}^1) = 0$. Thus, as a consequence, we prove the following

Theorem G. *Let G be a compact, connected and simply connected, simple Lie group, and $[c] \in H^2((G \rtimes G)_\bullet, \mathcal{S}^1) \cong \mathbb{Z}$. Then the equivariant twisted K -theory group $K_{[c], G}^*(G)$ is endowed with a canonical ring structure*

$$K_{[c], G}^{i+d}(G) \otimes K_{[c], G}^{j+d}(G) \rightarrow K_{[c], G}^{i+j+d}(G),$$

where $d = \dim G$, in the sense that there is a canonical isomorphism of the rings when using any two 2-cocycles in $\check{Z}^2((G \rtimes G)_\bullet; \mathcal{S}^1)$ which are in the images of the transgression T_1 .

This paper is organized as follows. Section 2 is devoted to preliminaries. In particular, we introduce generalized simplicial-categories and cohomology of generalized simplicial-spaces. In Section 3, we give the construction of the transgression maps and discuss their properties. Section 4 is devoted to the discussion of the ring structures of twisted equivariant K -theory groups.

Note that Freed-Hopkins-Teleman has proved a remarkable theorem that the equivariant twisted K -theory group of a compact connected Lie group admits a ring structure which is isomorphic to Verlinde algebra [24, 26]. Indeed the idea of using multiplicative cocycles (called equivariantly primitive in [25]) in constructing the product on twisted equivariant K -theory has been known in the community. See [25, 28, 14, 16] for instance. In [14, 16], there is a notion of “multiplicative bundle gerbes”, which seems to be equivalent to our “2-multiplicativity” except for the fact that “multiplicative bundle gerbes” in the sense of Carey et. al. [14, 16] are gerbes over G , while our gerbes are over $[G/G]$ (i.e. G -equivariant gerbes). In [16], it has been addressed the issue that the multiplicative property is related to the ring structure on the twisted K -theory groups. The idea of considering the ring structure of K -theory groups twisted by classes arising from the transgression $H^4(BG, \mathbb{Z}) \rightarrow H_G^3(G, \mathbb{Z})$ is also standard [25, 28, 16]. The ring structures on twisted K -theory of orbifolds have been studied independently by Adem–Ruan–Zhang using a different method [1].

However, we would like to stress that the main purpose of our paper is to emphasize the importance of the use of techniques of groupoid C^* -algebras and KK -theory in the

study of theory of twisted K -theory. This is an advantage of working with twisted K -theory defined as the K -theory group of a certain C^* -algebra. We also aim to provide a different cohomological interpretation of the transgression maps and of 2-multiplicativity in a general framework.

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2 Preliminaries

2.1 General notations and definitions

Given any category \mathcal{C} (in particular any groupoid), the collection of objects is denoted by \mathcal{C}_0 and the collection of morphisms is denoted by \mathcal{C}_1 . We use Γ or $\Gamma \rightrightarrows \Gamma_0$ to denote a groupoid. As usual, Γ is identified with its set of arrows Γ_1 .

If $f : x \rightarrow y$ is a morphism, then x is called the source of f and is denoted by $s(f)$, and $y = t(f)$ is called the target of f . Hence the composition fg is defined if and only if $s(f) = t(g)$.

Given any $A \subset \mathcal{C}_1$, by A^y , A_x and A_x^y we denote $A \cap t^{-1}(y)$, $A \cap s^{-1}(x)$ and $A_x \cap A^y$, respectively.

For all $n \geq 1$, we denote by \mathcal{C}_n the set of composable n -tuples, i.e.

$$\mathcal{C}_n = \{(f_1, \dots, f_n) \mid s(f_1) = t(f_2), \dots, s(f_{n-1}) = t(f_n)\}.$$

Let Γ be a groupoid and $f : M \rightarrow \Gamma_0$ be a map. We will denote by $f^*\Gamma$, or by $\Gamma[M]$ if there is no ambiguity, *the pull-back groupoid* defined by

$$\Gamma[M]_0 = M, \quad \Gamma[M]_1 = \{(x, y, g) \in M \times M \times \Gamma \mid f(x) = t(g), f(y) = s(g)\}$$

with source and target maps $t(x, y, g) = x$, $s(x, y, g) = y$, product $(x, y, g)(y, z, h) = (x, z, gh)$ and inverse $(x, y, g)^{-1} = (y, x, g^{-1})$. In other words, $\Gamma[M]$ is the fibered product of the pair groupoid $M \times M$ and Γ over $\Gamma_0 \times \Gamma_0$.

Let us recall the definition of an action of a groupoid. By definition, a *right action* of a groupoid Γ on a space Z is given by

- (i) a map $J : Z \rightarrow \Gamma_0$, called the momentum map;
- (ii) a map $Z \times_{\Gamma_0} \Gamma := \{(z, g) \in Z \times \Gamma \mid J(z) = t(g)\} \rightarrow Z$, denoted by $(z, g) \mapsto zg$, satisfying $J(zg) = s(g)$, $z(gh) = (zg)h$ and $z \cdot J(z) = z$ whenever $J(z) = t(g)$ and $s(g) = t(h)$.

Then the *transformation groupoid* (also called *the crossed product groupoid*) $Z \rtimes \Gamma$ is defined by $(Z \rtimes \Gamma)_0 = Z$, and $(Z \rtimes \Gamma)_1 = Z \times_{\Gamma_0} \Gamma$, while the source map, target map and the product are $s(z, g) = zg$, $t(z, g) = z$, $(z, g)(zg, h) = (z, gh)$.

A groupoid Γ is said to be *proper* if $(t, s) : \Gamma \rightarrow \Gamma_0 \times \Gamma_0$ is a proper map. An action of Γ on Z is proper if $Z \rtimes \Gamma$ is a proper groupoid.

Definition 2.1 Let $N \rightrightarrows N_0$ and $\Gamma \rightrightarrows \Gamma_0$ be groupoids. We say that Γ acts on N by automorphisms if both N and N_0 are right Γ -spaces and the actions are compatible in the following sense

- the source and target maps $s, t : N \rightarrow N_0$ are Γ -equivariant,
- $x^g y^g = (xy)^g$ for all $(x, y, g) \in N \times N \times \Gamma$ whenever either side makes sense. Here x^g denotes the action of $g \in \Gamma$ on $x \in N$.

Given such a pair of groupoids (N, Γ) , one can form the *semi-direct product groupoid* $N \rtimes \Gamma$, where the unit space is N_0 , the space of morphisms is

$$(N \rtimes \Gamma)_1 = \{(x, g) \in N \times \Gamma \mid x^g \text{ makes sense}\},$$

the target, the source, the multiplication and the inverse are defined by

$$t(x, g) = t(x), \quad s(x, g) = s(x^g), \quad (x, g)(y, h) = (xy^{(g^{-1})}, gh), \quad \text{and} \quad (x, g)^{-1} = ((x^g)^{-1}, g^{-1}).$$

Definition 2.2 A crossed module is a groupoid morphism

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & \Gamma \\ \Downarrow & & \Downarrow \\ N_0 & \xrightarrow{=} & \Gamma_0 \end{array}$$

where $N \rightrightarrows N_0$ is a bundle of groups, together with an action of Γ on N by automorphisms such that

- (i) $\varphi(x^g) = \varphi(x)^g$ for all $x \in N$ and $g \in \Gamma$ such that x^g makes sense;
- (ii) $x^{\varphi(y)} = x^y$ for all composable pairs $(x, y) \in N_2$.

Here $\varphi(x)^g := g^{-1}\varphi(x)g$ and $x^y := y^{-1}xy$. For short, a crossed module is denoted by $N \xrightarrow{\varphi} \Gamma$.

A standard example of crossed modules is the inertia groupoid. Let $\Gamma \rightrightarrows \Gamma_0$ be a groupoid and $S\Gamma = \{g \in \Gamma \mid s(g) = t(g)\}$ be the space of closed loops in Γ . Then the canonical inclusion $S\Gamma \rightarrow \Gamma$, together with the conjugation action of Γ on $S\Gamma$, forms a crossed module, where the crossed-product groupoid $S\Gamma \rtimes \Gamma$ is called the inertia groupoid and is denoted by $\Lambda\Gamma$.

Definition 2.3 Let $N \xrightarrow{\varphi} \Gamma$ and $N' \xrightarrow{\varphi'} \Gamma'$ be crossed modules. A crossed module morphism $\tau : (N \xrightarrow{\varphi} \Gamma) \rightarrow (N' \xrightarrow{\varphi'} \Gamma')$ is a commutative diagram of groupoid morphisms

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & \Gamma \\ \tau \downarrow & & \downarrow \tau \\ N' & \xrightarrow{\varphi'} & \Gamma' \end{array}$$

satisfying the condition

$$\tau(x^g) = \tau(x)^{\tau(g)}, \quad \text{for all compatible } x \in N, g \in \Gamma. \quad (4)$$

Given a crossed module $N \xrightarrow{\varphi} \Gamma$, since φ maps N to $S\Gamma$, we have a natural crossed module morphism from $N \xrightarrow{\varphi} \Gamma$ to $S\Gamma \rightarrow \Gamma$.

2.2 \mathcal{C} -spaces and sheaf cohomology

Let \mathcal{C} be a category. By a \mathcal{C} -space, we mean a contravariant functor from the category \mathcal{C} to the category of topological spaces. Similarly, one defines a \mathcal{C} -manifold. Consider a \mathcal{C} -space M_\bullet . Let \mathcal{C}_{M_\bullet} be the category whose objects are pairs (i, U) , with $i \in \mathcal{C}_0$ and U an open subset of M_i , such that morphisms from (i, U) to (j, V) consist of those $f \in \text{Hom}_{\mathcal{C}}(j, i)$ for which $\tilde{f}(U) \subset V$. By definition, an abelian presheaf on the \mathcal{C} -space M_\bullet is an abelian presheaf on the category \mathcal{C}_{M_\bullet} , i.e. a contravariant functor from the category \mathcal{C}_{M_\bullet} to the category of abelian groups. A presheaf \mathcal{A} on M_\bullet restricts to a presheaf \mathcal{A}_i on each space M_i . We say that \mathcal{A} is a sheaf if each \mathcal{A}_i is a sheaf.

More concretely, a sheaf \mathcal{A} on M_\bullet is given by a family $(\mathcal{A}^i)_{i \in \mathcal{C}_0}$ such that \mathcal{A}^i is a sheaf on M_i , together with restriction maps $\tilde{f}^* : \mathcal{A}^j(V) \rightarrow \mathcal{A}^i(U)$, for each $\tilde{f} \in \text{Hom}_{\mathcal{C}_{M_\bullet}}((i, U), (j, V))$, satisfying the relation $(\tilde{f} \circ \tilde{g})^* = \tilde{g}^* \circ \tilde{f}^*$ [20]. In a similar fashion, one defines the notion of a sheaf over a \mathcal{C} -manifold. Note that a big sheaf over the site of all smooth manifolds naturally induces a sheaf on a \mathcal{C} -manifold. For instance, the sheaf of real-valued smooth functions \mathcal{R} , the sheaf of S^1 -valued smooth functions \mathcal{S}^1 , and the sheaf of q -forms Ω^q (for fixed q) are examples of such sheaves.

Assume that \mathcal{A} is a sheaf on a \mathcal{C} -space M_\bullet . In order to define cohomology groups $H^*(M_\bullet, \mathcal{A})$, one needs an extra structure on \mathcal{C} .

We say that a category \mathcal{C} is a generalized simplicial category if every object $k \in \mathcal{C}_0$ is labeled by an integer $\deg(k) \in \mathbb{N}$ (in other words, there is a functor from the category \mathcal{C} to the groupoid $\mathbb{N} \times \mathbb{N} \rightrightarrows \mathbb{N}$), and moreover it is endowed with a set $A \subset \mathcal{C}_1$ and $\varepsilon : A \rightarrow \mathbb{Z}$ satisfying

- (i) A_k is finite for all $k \in \mathcal{C}_0$;
- (ii) for all $f \in A$, $\deg(f) = 1$, where $\deg(f) = \deg(t(f)) - \deg(s(f))$;
- (iii) for all $f \in \mathcal{C}_1$,

$$\sum_{\substack{f' \circ f'' = f \\ f', f'' \in A}} \varepsilon(f') \varepsilon(f'') = 0.$$

Note that the sum in (iii) is finite due to (i).

Given a generalized simplicial category \mathcal{C} , a \mathcal{C} -space M_\bullet and a sheaf \mathcal{A} over M_\bullet , let

$$C^n(M_\bullet; \mathcal{A}) = \oplus_{\deg k = n} \mathcal{A}(M_k).$$

Then $C^*(M_\bullet; \mathcal{A})$ is endowed with a degree 1 differential

$$\partial \omega = \sum_{f \in A_k} \varepsilon(f) \tilde{f}^* \omega,$$

$\forall \omega \in \mathcal{A}(M_k)$. It is simple to check that $\partial^2 = 0$. Then we obtain a cochain complex $(C^*(M_\bullet; \mathcal{A}), \partial)$. By $Z^*(M_\bullet; \mathcal{A})$ we denote its space of cocycles and $\mathbb{H}^*(M_\bullet; \mathcal{A})$ the cohomology group.

Now given a complex of sheaves over M_\bullet , bounded below: $\mathcal{A}_0 \xrightarrow{d} \mathcal{A}_1 \xrightarrow{d} \mathcal{A}_2 \xrightarrow{d} \dots$, then

$$C^{p,q}(M_\bullet; \mathcal{A}_\bullet) := C^p(M_\bullet; \mathcal{A}_q) = \oplus_{\deg k = p} \mathcal{A}_q(M_k)$$

is endowed with a double complex structure with differentials d and ∂ . We denote by δ the total differential $(-1)^p d + \partial$, and by $\mathbb{H}^*(M_\bullet; \mathcal{A}_\bullet)$ its hypercohomology groups.

For a sheaf \mathcal{A} on M_\bullet , let \mathcal{A}_\bullet be a complex of sheaves over M_\bullet which is an injective resolution of \mathcal{A} . Then $H^*(M_\bullet; \mathcal{A})$ is defined to be $\mathbb{H}^*(M_\bullet; \mathcal{A}_\bullet)$, called the sheaf cohomology group of M_\bullet with coefficients in \mathcal{A} .

A particular case is the following: if M_\bullet is a \mathcal{C} -manifold and $\mathcal{A}_q = \Omega^q$: $\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$ is the de Rham complex of sheaves, the group $\mathbb{H}^*(M_\bullet; \Omega^\bullet)$ is called the de Rham cohomology of M_\bullet and is denoted by $H_{dR}^*(M_\bullet)$. It is isomorphic to $H^*(M_\bullet; \mathbb{R})$.

2.3 Simplicial spaces

Recall that the simplicial category, denoted by Δ , has as objects the set of non-negative integers, and $\text{Hom}_\Delta(k, k')$ is the set of non-decreasing maps from $[k]$ to $[k']$, where $[k] = \{0, \dots, k\}$. A Δ -space is thus called a simplicial (topological) space, and a Δ -manifold is a simplicial manifold.

In a down-to-earth term, a simplicial space is given by a sequence $M_\bullet = (M_n)_{n \in \mathbb{N}}$ of spaces, and for each $f \in \text{Hom}_\Delta(k, n)$, we are given a map (called face or degeneracy map depending which of k and n is larger) $\tilde{f} : M_n \rightarrow M_k$ such that $\tilde{f} \circ \tilde{g} = \tilde{g} \circ f$.

Similarly, denote by $\bar{\Delta}$ the category obtained from Δ by identifying $f : [k] \rightarrow [n]$ with $f' : [k] \rightarrow [n]$ whenever both f and f' are constant. We will call $\bar{\Delta}$ the reduced simplicial category.

A groupoid naturally gives rise to a simplicial space. To see this, consider the pair groupoid $[n] \times [n] \rightrightarrows [n]$. For a groupoid $\Gamma \rightrightarrows \Gamma_0$, let $\Gamma_n = \text{Hom}([n] \times [n], \Gamma)$ be the space of homomorphisms from the pair groupoid $[n] \times [n] \rightrightarrows [n]$ to Γ . Any $f \in \text{Hom}_\Delta(k, n)$ gives rise to a groupoid homomorphism from $[k] \times [k] \rightrightarrows [k]$ to $[n] \times [n] \rightrightarrows [n]$, again denoted by f . It thus, in turn, induces a map $\tilde{f} : \Gamma_n (= \text{Hom}([n] \times [n], \Gamma)) \rightarrow \Gamma_k (= \text{Hom}([k] \times [k], \Gamma))$, which is the “face/degeneracy” map. Note that Γ_n can be identified with the space of composable n -tuples: $\Gamma_n = \{(g_1, \dots, g_n) \mid g_1 \cdots g_n \text{ makes sense}\}$ since the groupoid $[n] \times [n] \rightrightarrows [n]$ is generated by elements $(i-1, i)$ ($1 \leq i \leq n$). Hence any groupoid morphism from $[n] \times [n] \rightrightarrows [n]$ to $\Gamma \rightrightarrows \Gamma_0$ is uniquely determined by the image of each element $(i-1, i)$, which is denoted by g_i ($1 \leq i \leq n$).

Moreover, the simplicial space structure descends to a reduced simplicial structure when the source and target maps coincide, i.e. when $\Gamma \rightrightarrows \Gamma_0$ is a bundle of groups.

Recall that the simplicial category Δ is equipped with a natural generalized simplicial category structure. The degree map is obviously the identity map $\Delta_0 \rightarrow \mathbb{N}$. For all $k \in \mathbb{N}$, let $\varepsilon_i^k : [k] \rightarrow [k+1]$ be the unique increasing map which omits i ($i = 0, \dots, k+1$):

$$\varepsilon_i^k(0) = 0, \dots, \varepsilon_i^k(i-1) = i-1, \varepsilon_i^k(i) = i+1, \dots, \varepsilon_i^k(k) = k+1.$$

We will omit the superscript k if there is no ambiguity. Let $\varepsilon(\varepsilon_i) = (-1)^i$. Then the pair (A, ε) , where $A_k = \{\varepsilon_i^k \mid i \in [k+1]\}$ is a generalized simplicial structure on Δ . For sheaf cohomology of simplicial manifolds, we refer the reader to [42, 22] for details.

Suppose now that \mathcal{C} and \mathcal{C}' are two generalized simplicial categories. Then the product $\mathcal{C}'' = \mathcal{C}' \times \mathcal{C}$ is naturally a generalized simplicial category, where $\deg(k, l) = \deg(k) + \deg(l)$, $A'' = A' \times \{1\} \cup \{1\} \times A$ and $\varepsilon(f, 1) = \varepsilon(f)$ for all $(f, 1) \in A_k' \times C_l$, $\varepsilon(1, g) = (-1)^{\deg k} \varepsilon(g)$ for all $(1, g) \in C_k' \times A_l$.

In particular, $\Delta \times \Delta$ is a generalized simplicial category. More precisely, if $\partial' = \sum_{i=0}^{k+1} (-1)^i \tilde{\varepsilon}_i^*$ is the differential with respect to the first simplicial structure (as above $\varepsilon_i' : [k] \rightarrow [k+1]$ is the increasing map that omits i) and $\partial = \sum_{i=0}^{l+1} (-1)^i \tilde{\varepsilon}_i^*$ is the differential with respect to the second simplicial structure, then $\underline{\partial} = \partial' + (-1)^k \partial$ is the differential for the bi-simplicial structure.

2.4 \mathbb{F} -spaces

Let us now consider an extension of $\bar{\Delta}$. Denote by \mathbb{F} the category with the same objects as Δ , but $\text{Hom}_{\mathbb{F}}(k, n)$ consists of $\text{Hom}(\mathbb{F}_k, \mathbb{F}_n)$, where \mathbb{F}_n is the free group on n generators. More concretely, any element of $\text{Hom}(\mathbb{F}_k, \mathbb{F}_n)$ is given by a k -tuple $w = (w_1, \dots, w_k)$ of words in $x_1^{\pm 1}, \dots, x_n^{\pm 1}$. Denoting by \hat{w} the corresponding element of $\text{Hom}_{\mathbb{F}}(k, n)$, we have the relation $\widehat{w' \circ w} = \hat{w}' \circ \hat{w}$, where $(w' \circ w)_i = w' \circ w_i$ consists of the word obtained from w_i by substituting each occurrence of x_j by w'_j .

To see that \mathbb{F} extends the category $\bar{\Delta}$, for any $f \in \text{Hom}_{\bar{\Delta}}(k, n)$, let $\bar{f} \in \text{Hom}_{\mathbb{F}}(k, n)$ be the morphism

$$\bar{f}(x_i) = x_{f(i-1)+1} \cdots x_{f(i)}, \quad i = 1, \dots, k$$

(with the convention $\bar{f}(x_i) = 1$ if $f(i-1) = f(i)$). One immediately checks that $\overline{f \circ g} = \bar{f} \circ \bar{g}$. Another way to explain the inclusion $\bar{\Delta} \subset \mathbb{F}$ is as follows. Any $f \in \text{Hom}_{\bar{\Delta}}(k, n)$ gives rise to a groupoid homomorphism from $[k] \times [k] \rightrightarrows [k]$ to $[n] \times [n] \rightrightarrows [n]$, again denoted by f . Let $\iota : [n] \times [n] \rightarrow \mathbb{F}_n$ be the unique groupoid morphism such that $(i-1, i)$ maps to x_i . Then \bar{f} is the unique group homomorphism such that the diagram

$$\begin{array}{ccc} [k] \times [k] & \xrightarrow{\iota} & \mathbb{F}_k \\ f \downarrow & & \downarrow \bar{f} \\ [n] \times [n] & \xrightarrow{\iota} & \mathbb{F}_n \end{array}$$

commutes.

As above, a \mathbb{F} -(topological) space is a contravariant functor from \mathbb{F} to the category of topological spaces. If G is a topological group, then we obtain an associated \mathbb{F} -space by setting $G_n = \text{Hom}(\mathbb{F}_n, G) (\cong G^n)$. In particular, since \mathbb{F} extends the category $\bar{\Delta}$, $G_{\bullet} = (G_n)_{n \in \mathbb{N}}$ is a reduced simplicial space and therefore a simplicial space. The simplicial structure can be seen, as in Section 2.3, by considering G as a groupoid.

2.5 $\mathbb{F}\Delta$ -spaces

We now introduce a category $\mathbb{F}\Delta$. Objects are pairs $(k, l) \in \mathbb{N}^2$. To describe morphisms, let us introduce some notations: let $X_{k,l}$ be the groupoid $\mathbb{F}_k \times ([l] \times [l]) \rightrightarrows [l]$, the product of the free group \mathbb{F}_k with the pair groupoid $[l] \times [l] \rightrightarrows [l]$. Then we define $\text{Hom}((k, l), (k', l'))$ as the set of groupoid morphisms $f : X_{k,l} \rightarrow X_{k',l'}$ such that the restriction of f to the unit space, again denoted by $f : [l] \rightarrow [l']$, is a nondecreasing function. In particular, for $k = 0$ we recover the simplicial category Δ and for $l = 0$ we recover the category \mathbb{F} . We also note that the sub-category of $\mathbb{F}\Delta$ consisting of morphisms $f : X_{k,l} \rightarrow X_{k',l'}$ of the form $f = (f_1, f_2)$, where $f_1 : \mathbb{F}_k \rightarrow \mathbb{F}_{k'}$ is a group morphism and $f_2 : [l] \times [l] \rightarrow [l'] \times [l']$ is a groupoid morphism whose restriction to the unit spaces $[l] \rightarrow [l']$ is nondecreasing, is exactly isomorphic to the product category $\mathbb{F} \times \Delta$.

To understand the category $\mathbb{F}\Delta$ in a more concrete way, consider the following arrows of the groupoid $X_{k,l}$:

$$\tilde{a} = (a, 0, 0), \quad \gamma_i = (1, 0, i), \quad (5)$$

where $a \in \mathbb{F}_k$ and $i = 0, \dots, l$. They generate $X_{k,l}$ since any arrow in $X_{k,l}$ can be written in a unique way as

$$(a, i, j) = \gamma_i^{-1} \tilde{a} \gamma_j, \quad (6)$$

where $a \in \mathbb{F}_k$. Consider any morphism in $\text{Hom}_{\mathbb{F}\Delta}((k, l), (k', l'))$, whose restriction to the unit space is denoted by $f : [l] \rightarrow [l']$. Assume that under this morphism, we have $\tilde{a} \mapsto (\psi(a), f(0), f(0)) \in X_{k', l'}$ and $\gamma_i \mapsto (u_i, f(0), f(i)) \in X_{k', l'}$, where $\psi \in \text{Hom}(\mathbb{F}_k, \mathbb{F}_{k'})$, $f \in \text{Hom}_{\Delta}(l, l')$, and $u = (u_0, \dots, u_l) \in (\mathbb{F}_{k'})^{l+1}$. Thus

$$(a, i, j) = \gamma_i^{-1} \tilde{a} \gamma_j \mapsto (u_i^{-1} \psi(a) u_j, f(i), f(j)).$$

Note that the triple (ψ, u, f) is uniquely determined modulo the equivalence relation: $(\psi, u, f) \sim (\psi', u', f)$ if $\psi'(a) = \psi(a)^v$ and $u'_i = v^{-1} u_i$ for some $v \in \mathbb{F}_k$.

We summarize the above discussion in the following

Proposition 2.4 *$\text{Hom}_{\mathbb{F}\Delta}((k, l), (k', l'))$ can be identified with triples (ψ, u, f) , where $\psi \in \text{Hom}(\mathbb{F}_k, \mathbb{F}_{k'})$, $f \in \text{Hom}_{\Delta}(l, l')$, and $u = (u_0, \dots, u_l) \in (\mathbb{F}_{k'})^{l+1}$, modulo the equivalence relation $(\psi, u, f) \sim (\psi', u', f)$ if and only if $\psi'(a) = \psi(a)^v$ and $u'_i = v^{-1} u_i$ for some $v \in \mathbb{F}_k$.*

The composition law of morphisms is then $(\psi', u', f') \circ (\psi, u, f) = (\psi'', u'', f'')$, where $\psi'' = \psi' \circ \psi$, $f'' = f' \circ f$ and $u''_i = \psi'(u_i) u'_{f(i)}$.

2.6 Δ_2 -spaces

Next we define a category Δ_2 as follows: objects are pairs of integers $(k, l) \in \mathbb{N}^2$. $\text{Hom}_{\Delta_2}((k, l), (k', l'))$ consists of triples (a, b, c) such that $a \in \{\emptyset\} \cup \text{Hom}_{\Delta}(k, k')$, $b \in \{\emptyset\} \cup \text{Hom}_{\Delta}(l, l')$, $c \in \text{Hom}_{\Delta}(l, l')$, and either $a = \emptyset$ or $b = \emptyset$.

We define the composition as follows.

$$(a', \emptyset, c') \circ (a, \emptyset, c) = (a' \circ a, \emptyset, c' \circ c), (a', \emptyset, c') \circ (\emptyset, b, c) = (\emptyset, a' \circ b, c' \circ c), (\emptyset, b', c') \circ (a, b, c) = (\emptyset, b' \circ c, c' \circ c).$$

The associativity can be checked easily and is left to the reader.

It is clear that the bi-simplicial category $\Delta \times \Delta$ embeds into Δ_2 by $(a, c) \in \text{Hom}_{\Delta}(k, k') \times \text{Hom}_{\Delta}(l, l') \cong \text{Hom}_{\Delta \times \Delta}((k, k'), (l, l')) \mapsto (a, \emptyset, c) \in \text{Hom}_{\Delta_2}((k, k'), (l, l'))$.

Let us now define a category $\bar{\Delta}_2$, which has the same objects as Δ_2 , and whose morphisms are obtained from morphisms of Δ_2 by identifying (\emptyset, b, c) , (\emptyset, b', c) , (a, \emptyset, c) , (a', \emptyset, c) whenever a, a', b and b' are constant functions. The resulting element is denoted by $0_{(k, l), (k', l'), c}$, or simply by 0_c if there is no ambiguity. One checks directly that this definition makes sense and that $0_{c'} \circ (a, b, c) = 0_{c' \circ c}$, $(a', b', c') \circ 0_c = 0_{c' \circ c}$.

The category $\bar{\Delta}_2$ embeds into $\mathbb{F}\Delta$ as a subcategory. Indeed, to a triple $(a, b, c) \in \text{Hom}_{\Delta_2}((k, l), (k', l'))$ one associates a triple $F(a, b, c) = (\psi_a, u_b, c) \in \text{Hom}_{\mathbb{F}\Delta}((k, l), (k', l'))$ as follows. Denote by x_i the generators of \mathbb{F}_k , and let $y_i = x_1 \cdots x_i$, with the convention $y_0 = 1$. Let

$$\psi_a(y_i) = y_{a(0)}^{-1} y_{a(i)} \text{ and } (u_b)_i = y_{b(i)}, \quad (7)$$

where, by convention, $a(i) = 0$ if $a = \emptyset$. Then $(a, b, c) \mapsto (\psi_a, u_b, c)$ is injective, and a simple calculation shows that $F((a', b', c') \circ (a, b, c)) = F(a', b', c') \circ F(a, b, c)$.

Note also that $\bar{\Delta} \times \Delta \subset \bar{\Delta}_2$ by $(a, c) \mapsto (a, \emptyset, c)$.

The above discussion can be summarized by the following diagram, where all maps are embeddings except for the two horizontal arrows on the left:

$$\begin{array}{ccccc} \Delta \times \Delta & \longrightarrow & \bar{\Delta} \times \Delta & \longrightarrow & \mathbb{F} \times \Delta \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_2 & \longrightarrow & \bar{\Delta}_2 & \longrightarrow & \mathbb{F}\Delta \end{array}$$

2.7 The $\mathbb{F}\Delta$ - and bi-simplicial spaces associated to a crossed module

We show that a crossed module $N \xrightarrow{\varphi} \Gamma$ naturally gives rise to a $\mathbb{F}\Delta$ -space. Recall that every groupoid Γ gives rise to a crossed module $S\Gamma \rightarrow \Gamma$, with Γ acting on $S\Gamma$ by conjugations (see Section 2.1). Let $(N \rtimes \Gamma)_{k,l}$ be the space of morphisms of crossed modules from $(SX_{k,l} \rightarrow X_{k,l})$ to $(N \rightarrow \Gamma)$. Since a groupoid morphism $f : X_{k,l} \rightarrow X_{k',l'}$ induces a crossed module morphism from $(SX_{k,l} \rightarrow X_{k,l})$ to $(SX_{k',l'} \rightarrow X_{k',l'})$, we obtain a map $\tilde{f} : (N \rtimes \Gamma)_{k',l'} \rightarrow (N \rtimes \Gamma)_{k,l}$. Hence, $(N \rtimes \Gamma)_{\bullet\bullet} = ((N \rtimes \Gamma)_{k,l})_{(k,l) \in \mathbb{N}^2}$ is endowed with a structure of $\mathbb{F}\Delta$ -space.

To see this $\mathbb{F}\Delta$ -space $(N \rtimes \Gamma)_{\bullet\bullet}$ in down-to-earth terms, let $\tau \in (N \rtimes \Gamma)_{k,l}$ be an arbitrary element, which corresponds to a commutative diagram of groupoid morphisms

$$\begin{array}{ccc} SX_{k,l} & \longrightarrow & X_{k,l} \\ \tau \downarrow & & \downarrow \tau \\ N & \xrightarrow{\varphi} & \Gamma \end{array}$$

satisfying Eq. (4). Consider \mathbb{F}_k as a subgroupoid of $SX_{k,l} \rightrightarrows [l]$ by identifying \mathbb{F}_k with $\mathbb{F}_k \times \{(0,0)\}$, and let $h : \mathbb{F}_k \rightarrow N$ be the restriction of $\tau : SX_{k,l} \rightarrow N$ to \mathbb{F}_k . And let $g_i = \tau(\gamma_i)$, $i = 0, \dots, l$, where γ_i is as in Eq. (5). Using Eq. (6), for elements in $SX_{k,l}$, the map τ is given by

$$\tau(a, i, i) = \tau((a, 0, 0)^{\gamma_i}) = h(a)^{g_i}, \quad \forall a \in \mathbb{F}_k, i = 0, \dots, l,$$

while for elements in $X_{k,l}$, the map τ is then given by

$$\tau(a, i, j) = \tau(\gamma_i^{-1} \tilde{a} \gamma_j) = g_i^{-1} \tau(\tilde{a}) g_j = g_i^{-1} \varphi(h(a)) g_j, \quad \forall a \in \mathbb{F}_k, i, j = 0, \dots, l.$$

Note that the pair (h, g) is not unique and it is uniquely determined modulo the equivalence relation $(h, g) \sim (h', g')$, if $h'(a) = h(a)^r$ and $g'_i = r^{-1} g_i$ for some $r \in \Gamma$. Also note that $\text{Hom}(\mathbb{F}_k, N)$ can be naturally identified with N_k by identifying $h \in \text{Hom}(\mathbb{F}_k, N)$ with $(h(w_1), \dots, h(w_k)) \in N_k$, where w_1, \dots, w_k are generators of \mathbb{F}_k . Hence it follows that $(N \rtimes \Gamma)_{k,l}$ is isomorphic, as a space, to the quotient of

$$\{(x_1, \dots, x_k; g_0, \dots, g_l) \in N^k \times \Gamma^{l+1} \mid t(x_i) = t(g_j) \forall i, j\} \quad (8)$$

modulo the equivalence relation

$$(x_1, \dots, x_k; g_0, \dots, g_l) \sim (r^{-1} x_1 r, \dots, r^{-1} x_k r; r^{-1} g_0, \dots, r^{-1} g_l).$$

A simple calculation shows that the $\mathbb{F}\Delta$ -structure on $(N \rtimes \Gamma)_{\bullet\bullet}$ is given by $(\psi, u, f) \cdot (h', g') = (h, g)$, where

$$h(a) = h'(\psi(a)) \text{ and } g_i = h'(u_i) g'_{f(i)}. \quad (9)$$

Here (ψ, u, f) is a triple defining a morphism in $\text{Hom}_{\mathbb{F}\Delta}((k, l), (k', l'))$ as in Proposition 2.4. Since any $\mathbb{F}\Delta$ -space is automatically a bi-simplicial space, $(N \rtimes \Gamma)_{k,l}$ is naturally a bi-simplicial space. On the other hand, for any fixed k , the groupoid Γ acts on the space N_k . Hence we obtain a simplicial space

$$\dots N_k \rtimes \Gamma_2 \rightrightarrows N_k \rtimes \Gamma_1 \rightrightarrows N_k \rtimes \Gamma_0, \quad (10)$$

where

$$N_k \rtimes \Gamma_l = \{(x_1, \dots, x_k; g_1, \dots, g_l) \in N^k \times \Gamma^l \mid t(x_1) = \dots = t(x_k) = t(g_1), g_1 \cdots g_l \text{ makes sense}\}.$$

Moreover, for any fixed l , we get a simplicial structure on $N_\bullet \rtimes \Gamma_l$ since N is a groupoid. In fact, $N_\bullet \rtimes \Gamma_\bullet$ is a bi-simplicial space. Introduce a map

$$\begin{aligned} \Phi : N_k \rtimes \Gamma_l &\longrightarrow (N \rtimes \Gamma)_{k,l} \\ (x_1, \dots, x_k; g_1, \dots, g_l) &\mapsto [(x_1, \dots, x_k; t(x_1), g_1, g_1 g_2, \dots, g_1 \cdots g_l)]. \end{aligned} \quad (11)$$

It is clear that Φ establishes an isomorphism between $N_k \rtimes \Gamma_l$ and $(N \rtimes \Gamma)_{k,l}$ as topological spaces. Thus it follows that $N_\bullet \rtimes \Gamma_\bullet$ inherits a Δ_2 -structure, which is the pull-back of the Δ_2 -structure on $(N \rtimes \Gamma)_{\bullet\bullet}$ via Φ . The proposition below describes some of the morphisms of this Δ_2 -structure, which are useful in the following sections.

Proposition 2.5 *For any $f \in \text{Hom}_{\Delta_2}((k, l), (k', l'))$, let $\tilde{f} : N_{k'} \rtimes \Gamma_{l'} \rightarrow N_k \rtimes \Gamma_l$, $(x'_1, \dots, x'_{k'}; g'_1, \dots, g'_{l'}) \mapsto (x_1, \dots, x_k; g_1, \dots, g_l)$, be its corresponding map. Then*

1. *if $f = (\emptyset, b, c)$, then $g_i = (x'_{b(i-1)+1} \cdots x'_{b(i)})^{g'_1 \cdots g'_{c(i-1)}} g'_{c(i-1)+1} \cdots g'_{c(i)}$ and $x_i = t(g'_{c(0)+1})$; and*
2. *if $f = (a, \emptyset, c)$, then $x_i = x'_{a(i-1)+1} \cdots x'_{a(i)}$ and $g_i = g'_{c(i-1)+1} \cdots g'_{c(i)}$.*

PROOF. This follows from an elementary verification, which is left to the reader. \square

The following is an immediate consequence of Proposition 2.5 (2).

Corollary 2.6 *The map Φ is an isomorphism of bi-simplicial spaces, where the bi-simplicial structure on the left-hand side is the standard one described above and the bi-simplicial structure on the right-hand side is induced from the Δ_2 -structure.*

2.8 Čech Cohomology

Let us now define Čech cohomology. We refer the reader to [42] for the particular case of $\mathcal{C} = \Delta$; proofs of assertions for the categories Δ^2 and Δ_2 are essentially the same and will be omitted. Given a category \mathcal{C} equipped with a generalized simplicial structure A , assume that we are given a sub-category \mathcal{C}' such that

- (i) \mathcal{C}' contains A ;
- (ii) \mathcal{C}'^k is finite for all $k \in \mathcal{C}_0$.

Note that in this case A^k is necessarily finite for all $k \in \mathcal{C}_0$; conversely, if A^k is finite for all $k \in \mathcal{C}_0$, then the sub-category generated by A satisfies (i) and (ii) above. For instance, in the case of $\mathcal{C} = \Delta$, one can take \mathcal{C}' to be the pre-simplicial category Δ' , i.e. $\text{Hom}_{\Delta'}(k, k')$ consists of (strictly) increasing maps $[k] \rightarrow [k']$. For $\mathcal{C} = \Delta_2$, \mathcal{C}' will be the set of degree ≥ 0 morphisms (recall that $\deg(f) = \deg(t(f)) - \deg(s(f))$). The reason why we define \mathcal{C}' this way is that we need morphisms f_σ (see Eq. (12)) to belong to \mathcal{C}' .

An *open cover* of a \mathcal{C} -space M_\bullet is a collection (\mathcal{U}^k) , indexed by $k \in \mathcal{C}_0$ such that $\mathcal{U}^k = (U_i^k)_{i \in I_k}$ is an open cover of the topological space M_k . A \mathcal{C}' -cover is an open cover, together with a \mathcal{C}' -structure on I_\bullet such that for all $f \in \mathcal{C}'_1$ and all $i \in I_{t(f)}$, $\tilde{f}(U_i) \subset U_{\tilde{f}(i)}$.

Given any open cover, there is a canonical \mathcal{C}' -cover which is finer. Indeed, let Λ_k be the set of families $\lambda = (\lambda_f)_{f \in \mathcal{C}'_k}$ such that $\lambda_f \in I_{s(f)}$. Let $V_\lambda^k = \cap_{f \in \mathcal{C}'_k} \tilde{f}^{-1}(U_{\lambda_f}^{s(f)})$. This is an open subset of M_k since it is the intersection of finitely many open subsets. Moreover, Λ_\bullet is endowed with a \mathcal{C}' -structure, by $(\tilde{h}\lambda)_g = \lambda_{h \circ g}$, $\forall h \in \mathcal{C}'_1$, and it is straightforward to check that the cover $(\sigma\mathcal{U}^k)$ defined by $\sigma\mathcal{U}^k := (V_\lambda^k)_{\lambda \in \Lambda_k}$, is a \mathcal{C}' -cover, called the \mathcal{C}' -refinement of (\mathcal{U}^k) .

Now given a \mathcal{C}' -cover (\mathcal{U}^k) , let $M'_k = \coprod_{i \in I_k} U_i^k$. Then M'_\bullet is endowed with a \mathcal{C}' -structure. Moreover, any sheaf \mathcal{A} on M_\bullet induces a sheaf on M'_\bullet , again denoted by \mathcal{A} , by $\mathcal{A}(U) = \prod_{i \in I_k} \mathcal{A}(U \cap U_i^k)$ (U is any open subset of M'_k).

Since \mathcal{C}' is a generalized simplicial category, one can define $C^*(M'_\bullet; \mathcal{A})$, $Z^*(M'_\bullet; \mathcal{A})$ and $\mathbb{H}^*(M'_\bullet; \mathcal{A})$ as in Section 2.2. These groups will be denoted by $C^*(\mathcal{U}; \mathcal{A})$, $Z^*(\mathcal{U}; \mathcal{A})$ and $H^*(\mathcal{U}; \mathcal{A})$ respectively. Then the Čech cohomology groups $\check{H}^*(M_\bullet; \mathcal{A})$ are, by definition, the inductive limit of the groups $H^*(\mathcal{U}; \mathcal{A})$, when \mathcal{U} runs over all \mathcal{C}' -covers of M_\bullet (the inductive limit being taken in the generalized sense of limits of functors, since a \mathcal{C}' -cover may be refined to another by several different ways).

Note that the Čech cohomology groups do not depend on the choice of \mathcal{C}' satisfying (i) and (ii) above. Indeed, let \mathcal{C}'' be the category generated by A . Since any \mathcal{C}' -cover is a \mathcal{C}'' -cover, and since any \mathcal{C}'' -cover admits a \mathcal{C}' -cover which is finer, it follows easily that the Čech cohomology groups defined using \mathcal{C}' coincide with those defined using \mathcal{C}'' .

When $\mathcal{C} = \Delta$, Δ^2 or Δ_2 , the Čech cohomology groups can also be seen as the cohomology groups of a “canonical Čech complex” $\check{C}^*(M_\bullet; \mathcal{A})$, which is, by definition, the inductive limit of the complexes $C^*(\sigma\mathcal{U}; \mathcal{A})$, where \mathcal{U} runs over all covers of the form $\mathcal{U}^k = (U_x^k)_{x \in M_k}$, and U_x^k is an open neighborhood of x ; the cover \mathcal{U}' is said to be finer than \mathcal{U} if $(U'_x)^k \subset U_x^k$ for all k and $x \in M_k$ (see [42] for details). In the sequel, Čech cochains (resp. Čech cocycles) should be understood in the above sense.

3 The transgression maps

The purpose of this section is to show that there is a natural transgression map on the level of cochains for the cohomology of a Δ_2 -space. As a consequence, we prove that for a crossed module $N \xrightarrow{\varphi} \Gamma$ there exist transgression maps

$$T : \mathbb{H}^*(\Gamma_\bullet; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^*((N \rtimes \Gamma)_{\bullet\bullet}; \mathcal{A}_\bullet),$$

and

$$T_1 : \mathbb{H}^*(\Gamma_\bullet; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^{*-1}((N \rtimes \Gamma)_{\bullet\bullet}; \mathcal{A}_\bullet),$$

and similarly for Čech cohomology. Throughout this section, $M_{\bullet\bullet}$ denotes a Δ_2 -space.

3.1 Construction of the transgression maps

For any fixed $k \in \mathbb{N}$, consider the restriction of the category Δ_2 to the objects of the form (k, l) ($l \in \mathbb{N}$), and to morphisms of the form $\text{Id} \times (f \times f) : \mathbb{F}_k \times [l] \times [l] \rightarrow \mathbb{F}_k \times [l'] \times [l']$, where $f : [l] \rightarrow [l']$ is non-decreasing. This category is isomorphic to Δ . Hence we obtain a simplicial space $M_{k,\bullet} = (M_{k,l})_{l \in \mathbb{N}}$. Similarly, $M_{\bullet,k} = (M_{l,k})_{l \in \mathbb{N}}$ is also a simplicial space.

Let $\mathcal{A}_0 \xrightarrow{d} \mathcal{A}_1 \xrightarrow{d} \dots$ be a complex of abelian sheaves over $M_{\bullet\bullet}$, and $C^*(M_{\bullet\bullet}; \mathcal{A}_\bullet)$ (resp. $C^*(M_{0,\bullet}; \mathcal{A}_\bullet)$) its associated cochain complex on $M_{\bullet\bullet}$ (resp. $M_{0,\bullet}$) as in Section 2.2. The main goal of this section is to construct chain maps $T : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^*(M_{\bullet\bullet}; \mathcal{A}_\bullet)$

and $T_1 : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^{*-1}(M_{1,\bullet}; \mathcal{A}_\bullet)$. Thus we construct natural transgression maps $T : \mathbb{H}^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^*(M_{\bullet,\bullet}; \mathcal{A}_\bullet)$, and $T_1 : \mathbb{H}^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^{*-1}(M_{1,\bullet}; \mathcal{A}_\bullet)$. As a consequence, for an abelian sheaf \mathcal{A} over $M_{\bullet,\bullet}$, we obtain transgression maps $T : H^*(M_{0,\bullet}; \mathcal{A}) \rightarrow H^*(M_{\bullet,\bullet}; \mathcal{A})$ and $T_1 : H^*(M_{0,\bullet}; \mathcal{A}) \rightarrow H^{*-1}(M_{1,\bullet}; \mathcal{A})$. Similarly, there are chain maps $T : \check{C}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{C}^*(M_{\bullet,\bullet}; \mathcal{A})$ and $T_1 : \check{C}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{C}^{*-1}(M_{1,\bullet}; \mathcal{A})$, which induce transgression maps for Čech cohomology $T : \check{H}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{H}^*(M_{\bullet,\bullet}; \mathcal{A})$ and $T_1 : \check{H}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{H}^{*-1}(M_{1,\bullet}; \mathcal{A})$ as well.

We first give the construction for sheaf cohomology. First of all, we need to introduce some notations.

Recall that a (k, l) -shuffle is a permutation σ of $\{1, \dots, k+l\}$ such that $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+l)$. One can represent a (k, l) -shuffle by a sequence of balls, k of which being black and l of which being white. More precisely, $B_\sigma = \sigma(\{1, \dots, k\})$ and $W_\sigma = \sigma(\{k+1, \dots, k+l\})$. The signature of σ can be easily computed by the formula $\varepsilon(\sigma) = (-1)^{\sum_{i \leq k} \sigma(i) - i}$.

Denote by $S_{k,l}$ the set of (k, l) -shuffles. For any $\sigma \in S_{k,l}$, we define $f_\sigma \in \text{Hom}_{\Delta_2}((0, k+l), (k, l))$ by

$$f_\sigma = (\emptyset, b^\sigma, c^\sigma), \quad (12)$$

where b^σ is the map $[k+l] \rightarrow [k]$ given by $b^\sigma(i) = \#(B_\sigma \cap \{1, \dots, i\})$, and c^σ is the map $[k+l] \rightarrow [l]$ given by $c^\sigma(i) = \#(W_\sigma \cap \{1, \dots, i\})$, $i = 0, \dots, k+l$. Thus f_σ induces $\tilde{f}_\sigma : M_{k,l} \rightarrow M_{0,k+l}$. Therefore, we obtain a map $\tilde{f}_\sigma^* : \mathcal{A}^q(M_{0,k+l}) \rightarrow \mathcal{A}^q(M_{k,l})$. Taking the direct sum over all l and q , we obtain a map

$$\tilde{f}_\sigma^* : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^{*-k}(M_{k,\bullet}; \mathcal{A}_\bullet).$$

Set

$$T_k = \sum_{\sigma \in S_{k,l}} \varepsilon(\sigma) \tilde{f}_\sigma^* : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^{*-k}(M_{k,\bullet}; \mathcal{A}_\bullet)$$

(with $T_0 = \text{Id}$), and

$$T = \sum_{k \geq 0} T_k : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^*(M_{\bullet,\bullet}; \mathcal{A}_\bullet)$$

using the decomposition $C^*(M_{\bullet,\bullet}; \mathcal{A}_\bullet) = \bigoplus_{k \geq 0} C^{*-k}(M_{k,\bullet}; \mathcal{A}_\bullet)$.

For any fixed $k \geq 0$, by ∂ and ∂' we denote the differentials

$$\partial : C^{p,q}(M_{k,\bullet}, \mathcal{A}_\bullet)(= \mathcal{A}_q(M_{k,p})) \longrightarrow C^{p+1,q}(M_{k,\bullet}, \mathcal{A}_\bullet)(= \mathcal{A}_q(M_{k,p+1})),$$

and

$$\partial' : C^{p,q}(M_{\bullet,k}, \mathcal{A}_\bullet)(= \mathcal{A}_q(M_{p,k})) \longrightarrow C^{p+1,q}(M_{\bullet,k}, \mathcal{A}_\bullet)(= \mathcal{A}_q(M_{p+1,k})),$$

respectively, and by $\delta_k = (-1)^p d + \partial$, we denote the total differential of the double complex $C^{p,q}(M_{k,\bullet}; \mathcal{A}_\bullet)$. Note that $C^*(M_{\bullet,\bullet}; \mathcal{A}_\bullet) = \sum_{p+q+k=*} \mathcal{A}_q(M_{k,p})$ and the total differential is $(-1)^{k+p} d + (-1)^k \partial + \partial'$.

Lemma 3.1 *Assume that $M_{\bullet,\bullet}$ is a Δ_2 -space, and \mathcal{A}_\bullet is a complex of abelian sheaves over $M_{\bullet,\bullet}$. Then*

$$\partial' T_k = T_{k+1} \partial + (-1)^k \partial T_{k+1} \quad \text{and} \quad (13)$$

$$\partial' T_k = T_{k+1} \delta_0 + (-1)^k \delta_{k+1} T_{k+1}, \quad (14)$$

where both sides are maps from $C^*(M_{0,\bullet}; \mathcal{A}_\bullet)$ to $C^{*-k}(M_{k+1,\bullet}; \mathcal{A}_\bullet)$.

PROOF. Let us first show that Eq. (14) follows from Eq. (13). For any $\omega \in C^{k+l,q}(M_{0,\bullet}, \mathcal{A}_\bullet)$, we have

$$\begin{aligned} T_{k+1}\delta_0\omega + (-1)^k\delta_{k+1}T_{k+1}\omega &= T_{k+1}((-1)^{k+l}d + \partial)\omega + (-1)^k((-1)^{l-1}d + \partial)T_{k+1}\omega \\ &= T_{k+1}\partial\omega + (-1)^k\partial T_{k+1}\omega. \end{aligned}$$

Here we have used the fact that T_{k+1} commutes with d since T_{k+1} is a summation of pull-back maps.

Now we prove Eq. (13). For any $\omega \in C^{k+l,q}(M_{0,\bullet}, \mathcal{A}_\bullet)$,

$$T_{k+1}\partial\omega = \sum_{\sigma \in S_{k+1,l}} \sum_{j=0}^{k+l+1} (-1)^j \varepsilon(\sigma) f_\sigma^* \tilde{\varepsilon}_j^* \omega.$$

In the sum above, we distinguish three cases:

- 1) $(j = 0, \sigma^{-1}(1) \leq k+1)^1$ or $(j = p, \sigma^{-1}(k+l+1) \leq k+1)^2$ or $(1 \leq j \leq k+l, \sigma^{-1}(j) \leq k+1, \sigma^{-1}(j+1) \leq k+1)$;
- 2) $(j = 0, \sigma^{-1}(1) \geq k+2)^3$, $(j = k+l+1, \sigma^{-1}(k+l+1) \geq k+2)^4$ or $(1 \leq j \leq k+l, \sigma^{-1}(j) \geq k+2, \sigma^{-1}(j+1) \geq k+2)$;
- 3) $1 \leq j \leq k+l$ and

(a) either $\sigma^{-1}(j) \leq k+1$ and $\sigma^{-1}(j+1) \geq k+2$

(b) or $\sigma^{-1}(j) \geq k+2$ and $\sigma^{-1}(j+1) \leq k+1$.

We show that the terms in 1) are equal to $\partial' T_k \omega$, the terms in 2) are equal to $(-1)^{k+1} \partial T_{k+1} \omega$ and the terms in 3a) cancel out with those in 3b).

Let us examine the terms in 1). We have

$$\begin{aligned} \partial' T_k \omega &= \sum_{m=0}^{k+1} (-1)^m \tilde{\varepsilon}_m^* T_k \omega \\ &= \sum_{m=0}^{k+1} \sum_{\tau \in S_{k,l}} (-1)^m \varepsilon(\tau) \tilde{\varepsilon}_m^* f_\tau^* \omega. \end{aligned}$$

Given (j, σ) as in 1), we define (m, τ) as follows: $m = \sigma^{-1}(j)$, with the convention $\sigma^{-1}(0) = 0$, and τ is uniquely determined by the equation

$$\varepsilon_{j \circ \tau} = \sigma \circ \varepsilon_m.$$

In other words, if the shuffle σ is represented by a sequence of $p = k+l+1$ balls, $k+1$ of which being black and l of which being white, then τ is obtained from σ by removing the j -th one (which is black).

We need to check the following equalities:

- (i) $(-1)^m \varepsilon(\tau) = (-1)^j \varepsilon(\sigma)$, and

¹thus, $\sigma^{-1}(1) = 1$

²thus, $\sigma^{-1}(k+l+1) = k+1$

³thus, $\sigma^{-1}(1) = k+2$

⁴thus, $\sigma^{-1}(k+l+1) = k+l+1$

$$(ii) \quad \tilde{f}_\sigma^* \tilde{\varepsilon}_j^* \omega = \tilde{\varepsilon}'_m \tilde{f}_\tau^* \omega.$$

To show (i), let $p = k + l + 1$ and $\sigma_{j,p}$ the circular permutation $(j, j+1, \dots, p)$. Then $\varepsilon_p \circ \tau = \sigma_{j,p}^{-1} \circ \varepsilon_j \circ \tau = \sigma_{j,p}^{-1} \circ \sigma \circ \varepsilon_m = (\sigma_{j,p}^{-1} \circ \sigma \circ \sigma_{m,p}) \circ \varepsilon_p$. Thus $\varepsilon(\tau) = \varepsilon(\sigma_{j,p}^{-1} \circ \sigma \circ \varepsilon_{m,p}) = (-1)^{j-m} \varepsilon(\sigma)$.

To show (ii), it suffices to prove that $f_{\sigma \circ \varepsilon_j} = \varepsilon'_m \circ f_\tau$. Now

$$\begin{aligned} f_{\sigma \circ \varepsilon_j} &= (0, b_\sigma, c_\sigma) \circ (\text{Id}, 0, \varepsilon_j) = (0, b_{\sigma \circ \varepsilon_j}, c_{\sigma \circ \varepsilon_j}), \quad \text{and} \\ \varepsilon'_m \circ f_\tau &= (\varepsilon_m, 0, \text{Id}) \circ (0, b_\tau, c_\tau) = (0, \varepsilon_m \circ b_\tau, c_\tau). \end{aligned}$$

Hence it remains to check that $b_{\sigma \circ \varepsilon_j} = \varepsilon_m \circ b_\tau$ and $c_{\sigma \circ \varepsilon_j} = c_\tau$, i.e. that

$$\begin{aligned} \#(B_\sigma \cap \{1, \dots, \varepsilon_j(i)\}) &= \varepsilon_m(\#(B_\tau \cap \{1, \dots, i\})), \quad \text{and} \\ \#(W_\sigma \cap \{1, \dots, \varepsilon_j(i)\}) &= \#(W_\tau \cap \{1, \dots, i\}), \end{aligned}$$

which is immediate from the description of B_τ and W_τ in terms of B_σ and W_σ .

The terms in 2) are treated in a similar fashion: define (m, τ) by $m = \sigma^{-1}(j)$ with the convention $\sigma^{-1}(0) = k+1$, and τ by the equation $\varepsilon_j \circ \tau = \sigma \circ \varepsilon_m$.

For the terms in 3), it is easy to check that the term corresponding to (j, σ) cancels out with $(j, \tau_{j,j+1} \circ \sigma)$, where $\tau_{j,j+1}$ is the transposition which exchanges j and $j+1$. \square

One can introduce transgression maps for Čech cohomology in a similar fashion. Namely, if $\mathcal{U} := (\mathcal{U}^{k,l})$ is a Δ'_2 -cover of $M_{\bullet,\bullet}$, then for any fixed k , $(\mathcal{U}^{k,l})$ is a pre-simplicial cover of $M_{k,\bullet}$, and $(\mathcal{U}^{l,k})$ is a pre-simplicial cover of $M_{\bullet,k}$. They are denoted, respectively, by $\mathcal{U}^{k,\bullet}$ and $\mathcal{U}^{\bullet,k}$.

Let $M'_{k,l} = \coprod_{i \in I_{k,l}} U_i^{k,l}$. Then $M'_{\bullet,\bullet}$ is endowed with a Δ'_2 -structure. Hence for any fixed k , $M'_{k,\bullet}$ and $M'_{\bullet,k}$ are pre-simplicial spaces. For any $\sigma \in S_{k,l}$, since $f_\sigma \in \text{Hom}_{\Delta_2}((0, k+l), (k, l))$, one has a map $\tilde{f}_\sigma : M'_{k,l} \rightarrow M'_{0,k+l}$. Thus $\tilde{f}_\sigma^* : \mathcal{A}(M'_{0,k+l}) \rightarrow \mathcal{A}(M'_{k,l})$. Set

$$T_k = \sum_{\sigma \in S_{k,l}} \varepsilon(\sigma) \tilde{f}_\sigma^* : \check{C}^*(\mathcal{U}^{0,\bullet}, \mathcal{A}) (= C^*(M'_{0,\bullet}; \mathcal{A})) \rightarrow \check{C}^*(\mathcal{U}^{k,\bullet}, \mathcal{A}) (= C^{*-k}(M'_{k,\bullet}; \mathcal{A}))$$

(with $T_0 = \text{Id}$), and

$$T = \sum_{k \geq 0} T_k : \check{C}^*(\mathcal{U}^{0,\bullet}, \mathcal{A}) (= C^*(M'_{0,\bullet}; \mathcal{A})) \rightarrow \check{C}^*(\mathcal{U}, \mathcal{A}) (= C^*(M'_{\bullet,\bullet}; \mathcal{A})),$$

using the decomposition $C^*(M'_{\bullet,\bullet}; \mathcal{A}) = \oplus_{k \geq 0} C^{*-k}(M'_{k,\bullet}; \mathcal{A})$. For any fixed $k \geq 0$, by ∂ and ∂' , we denote, respectively, the differentials

$$\partial : \check{C}^p(\mathcal{U}^{k,\bullet}, \mathcal{A}) (= \mathcal{A}(M'_{k,p})) \longrightarrow \check{C}^{p+1}(\mathcal{U}^{k,\bullet}, \mathcal{A}) (= \mathcal{A}(M'_{k,p+1})),$$

and

$$\partial' : \check{C}^p(\mathcal{U}^{\bullet,k}, \mathcal{A}) (= \mathcal{A}(M'_{p,k})) \longrightarrow \check{C}^{p+1}(\mathcal{U}^{\bullet,k}, \mathcal{A}) (= \mathcal{A}(M'_{p+1,k})).$$

Note that $\check{C}^*(\mathcal{U}, \mathcal{A}) = C^*(M'_{\bullet,\bullet}; \mathcal{A}) = \sum_{p+k=*} \mathcal{A}(M'_{k,p})$, and the total differential is $\partial + \partial'$.

Lemma 3.2 Assume that $M_{\bullet\bullet}$ is a Δ_2 -space. If \mathcal{A} is an abelian sheaf over $M_{\bullet\bullet}$ and $(\mathcal{U}^{k,l})$ is a Δ'_2 -cover of $M_{\bullet\bullet}$, then

$$\partial' T_k = T_{k+1} \partial + (-1)^k \partial T_{k+1}, \quad (15)$$

where both sides are maps from $\check{C}^*(\mathcal{U}^{0,\bullet}, \mathcal{A})$ to $\check{C}^{*-k}(\mathcal{U}^{k+1,\bullet}, \mathcal{A})$.

As an immediate consequence of Lemma 3.1-3.2, we obtain the following

Theorem 3.3 Assume that $M_{\bullet\bullet}$ is a Δ_2 -space.

1. If \mathcal{A}_\bullet is a complex of abelian sheaves over $M_{\bullet\bullet}$, then

$$T : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^*(M_{\bullet\bullet}; \mathcal{A}_\bullet) \quad (16)$$

is a chain map. Therefore it induces a morphism on the level of cohomology

$$T : \mathbb{H}^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^*(M_{\bullet\bullet}; \mathcal{A}_\bullet). \quad (17)$$

Similarly, given an abelian sheaf \mathcal{A} over $M_{\bullet\bullet}$, then

$$T : \check{C}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{C}^*(M_{\bullet\bullet}; \mathcal{A}) \quad (18)$$

is a chain map, and therefore it induces a morphism

$$T : \check{H}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{H}^*(M_{\bullet\bullet}; \mathcal{A}). \quad (19)$$

We call T the *total transgression map*.

Considering the special case $k = 1$ in Lemma 3.1 and Lemma 3.2, we immediately obtain the following

Theorem 3.4 Under the same hypothesis as in Theorem 3.3,

1. if \mathcal{A}_\bullet is a complex of abelian sheaves over $M_{\bullet\bullet}$, then the map

$$T_1 : C^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow C^{*-1}(M_{1,\bullet}; \mathcal{A}_\bullet) \quad (20)$$

satisfies the relations:

$$\begin{aligned} T_1 \partial + \partial T_1 &= 0, \\ T_1 \delta + \delta T_1 &= 0; \end{aligned} \quad (21)$$

and thus induces a morphism:

$$T_1 : \mathbb{H}^*(M_{0,\bullet}; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^{*-1}(M_{1,\bullet}; \mathcal{A}_\bullet).$$

2. Similarly, for any abelian sheaf \mathcal{A} over $M_{\bullet\bullet}$,

$$T_1 : \check{C}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{C}^{*-1}(M_{1,\bullet}; \mathcal{A}) \quad (22)$$

satisfies

$$T_1 \partial + \partial T_1 = 0, \quad (23)$$

and therefore we have a morphism:

$$T_1 : \check{H}^*(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{H}^{*-1}(M_{1,\bullet}; \mathcal{A}).$$

We call T_1 the *transgression map*.

3.2 Transgression maps for crossed modules

Let $N \xrightarrow{\phi} \Gamma$ be a crossed module. Denote by $M_{\bullet\bullet}$ its associated Δ_2 -space $(N \rtimes \Gamma)_{\bullet\bullet}$. Let $N_k = \{(x_1, \dots, x_k) \in N^k \mid s(x_1) = \dots = s(x_k) (= t(x_1) = \dots = t(x_k))\}$. Then N_k is endowed with a Γ -action, and the simplicial space associated to the crossed-product groupoid $N_k \rtimes \Gamma$ is precisely $M_{k,\bullet}$. As a special case, for $k = 0$ we get $M_{0,\bullet} = \Gamma_\bullet$ and for $k = 1$ we get $M_{1,\bullet} = (N \rtimes \Gamma)_\bullet$. When $N = S\Gamma$, $M_{1,\bullet} = (S\Gamma \rtimes \Gamma)_\bullet = (\Lambda\Gamma)_\bullet$. According to Theorem 3.3, for any complex of sheaves \mathcal{A}_\bullet over $(N \rtimes \Gamma)_{\bullet\bullet}$, we have the total transgression map

$$T : \mathbb{H}^*(\Gamma_\bullet; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^*((N \rtimes \Gamma)_{\bullet\bullet}; \mathcal{A}_\bullet).$$

In particular, we have

$$T : H_{dR}^*(\Gamma_\bullet) \rightarrow H_{dR}^*((N \rtimes \Gamma)_{\bullet\bullet}).$$

Recall that $T = \sum_k T_k$, where

$$T_k = \sum_{\sigma \in S_{k,l}} \varepsilon(\sigma) \tilde{f}_\sigma^* : C^{k+l}(\Gamma_\bullet; \mathcal{A}_\bullet) \longrightarrow C^l((N_k \rtimes \Gamma)_\bullet; \mathcal{A}_\bullet).$$

Using Proposition 2.5, we obtain the following explicit formula for \tilde{f}_σ :

Proposition 3.5 *Let $N \xrightarrow{\phi} \Gamma$ be a crossed module. Then $\forall \sigma \in S_{k,l}$, the map $\tilde{f}_\sigma : N_k \rtimes \Gamma_l \rightarrow \Gamma_{k+l}$ is given by*

$$\tilde{f}_\sigma(x_1, \dots, x_k; g_1, \dots, g_l) = (u_1, \dots, u_{k+l}), \quad (24)$$

where $u_i = g_{\sigma^{-1}(i)}$ if $\sigma^{-1}(i) \geq k+1$, and $u_i = \varphi \left(x_{\sigma^{-1}(i)}^{\prod_{\sigma^{-1}(j) > k, j < i} g_{\sigma^{-1}(j)}} \right)$ otherwise.

Example 3.6 If σ is the $(2, 2)$ -shuffle $(1, 3, 2, 4)$, then

$$\tilde{f}_\sigma(x_1, x_2, g_1, g_2) = (\varphi(x_1), g_1, \varphi(x_2)^{g_1}, g_2).$$

We also have the transgression map:

$$T_1 : \mathbb{H}^*(\Gamma_\bullet; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^{*-1}((N \rtimes \Gamma)_\bullet; \mathcal{A}_\bullet).$$

In particular, for any groupoid $\Gamma \rightrightarrows \Gamma_0$, we have the transgression map

$$T_1 : \mathbb{H}^*(\Gamma_\bullet; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^{*-1}((\Lambda\Gamma)_\bullet; \mathcal{A}_\bullet).$$

Proposition 3.7 *Let $N \xrightarrow{\phi} \Gamma$ be a crossed module. Then the transgression $T_1 : \mathbb{H}^*(\Gamma_\bullet; \mathcal{A}_\bullet) \rightarrow \mathbb{H}^{*-1}((N \rtimes \Gamma)_\bullet; \mathcal{A}_\bullet)$ is given, on the cochain level, by*

$$T_1 = \sum_{i=0}^{p-1} (-1)^i \tilde{f}_i^* : \mathcal{A}^q(\Gamma_p) \rightarrow \mathcal{A}^q(N \rtimes \Gamma_{p-1}),$$

where the map $\tilde{f}_i : N \rtimes \Gamma_{p-1} \rightarrow \Gamma_p$ is given by

$$\tilde{f}_i(x; g_1, \dots, g_{p-1}) = (g_1, \dots, g_i, \varphi(x)^{g_1 \cdots g_i}, g_{i+1}, \dots, g_{p-1}). \quad (25)$$

Remark 3.8 In the case of de Rham cohomology, the transgression $T_1 : \Omega^q(\Gamma_p) \rightarrow \Omega^q(N \rtimes \Gamma_{p-1})$ is defined by $T_1 = \sum_{i=0}^{p-1} (-1)^i \tilde{f}_i^*$, where \tilde{f}_i is given by Eq. (3). More generally, the map $T_k : \Omega^q(\Gamma_p) \rightarrow \Omega^q(N_k \rtimes \Gamma_{p-k})$ takes the form $T_k = \sum_{\sigma \in S_{k,p-k}} \varepsilon(\sigma) \tilde{f}_\sigma^*$, where \tilde{f}_σ is given by Eq. (2).

Remark 3.9 Consider the case that Γ is a Lie group G . Then ΛG is the transformation groupoid $G \rtimes G \rightrightarrows G$, where G acts on G by conjugation. On the cochain level, the total transgression map for the de Rham cohomology becomes

$$T : \Omega_{dR}^n(G_\bullet) \longrightarrow \Omega_{dR}^n((G \rtimes G)_{\bullet\bullet}), \quad (26)$$

and the transgression map is

$$T_1 : \Omega_{dR}^n(G_\bullet) \longrightarrow \Omega_{dR}^{n-1}((G \rtimes G)_\bullet). \quad (27)$$

On the other hand, there is a Bott-Shulman map [7, 8] $S(\mathfrak{g}^*)^G \rightarrow \Omega_{dR}^n(G_\bullet)$. We have the following

Conjecture The equivariant Bott-Shulman map Φ constructed by Jeffrey [32, 37] are compositions of the usual Bott-Shulman map with the total transgression map T , under which T_1 becomes Φ_1 in [32, 37].

More precisely, the composition of the Bott-Shulman map $S^p(\mathfrak{g}^*)^G \rightarrow \Omega_{dR}^{2p}(G_\bullet)$ with $T_k : \Omega_{dR}^{2p}(G_\bullet) \rightarrow \Omega_{dR}^{2p-k}((G \rtimes G^k)_\bullet)$ yields a map $S^p(\mathfrak{g}^*)^G \rightarrow \Omega_{dR}^{2p-k}((G \rtimes G^k)_\bullet)$, which should be the simplicial counterpart of the maps $\Phi_G^{(k)}$ in Section B8 of [37].

To end this subsection, we give a geometric description of the transgression map $T_1 : H^2(\Gamma_\bullet; \mathcal{S}^1) \rightarrow H^1(\Lambda\Gamma_\bullet; \mathcal{S}^1)$, which was introduced in [44].

Given a topological groupoid Γ and an element $\alpha \in H^2(\Gamma_\bullet; \mathcal{S}^1)$, represent α by an \mathcal{S}^1 -central extension

$$S^1 \rightarrow \tilde{\Gamma}' \rightarrow \Gamma'. \quad (28)$$

Let $L = \tilde{\Gamma}' \times_{S^1} \mathbb{C}$ be the complex line bundle associated to the S^1 -principal bundle $\tilde{\Gamma}' \rightarrow \Gamma'$, and $L' \rightarrow S\Gamma'$ its restriction to the subspace $S\Gamma'$. Then $L' \rightarrow S\Gamma'$ is a Γ' -equivariant bundle, where the Γ' -action is given by $\gamma \cdot \xi = \tilde{\gamma} \xi \tilde{\gamma}^{-1}$, for all compatible $\gamma \in \Gamma'$ and $\xi \in L'$. Here $\tilde{\gamma} \in \tilde{\Gamma}'$ is any lift of γ . Thus $L' \rightarrow S\Gamma'$ is an S^1 -bundle over $\Lambda\Gamma' \rightrightarrows S\Gamma'$, and hence it determines an element in $H^1(\Lambda\Gamma'; \mathcal{S}^1)$. Since Γ' is Morita equivalent to Γ , $\Lambda\Gamma'$ is Morita equivalent to $\Lambda\Gamma$. Thus we obtain an element in $H^1(\Lambda\Gamma; \mathcal{S}^1)$, which is denoted by $\tau(\alpha)$. It is a simple exercise to check that the element $\tau(\alpha)$ does not depend on the choice of the S^1 -central extension (28). Hence we obtain a map

$$\tau : H^2(\Gamma_\bullet; \mathcal{S}^1) \longrightarrow H^1(\Lambda\Gamma_\bullet; \mathcal{S}^1).$$

Proposition 3.10 *The map T_1 coincides with the opposite of the map τ .*

PROOF. Represent an element $\alpha \in H^2(\Gamma_\bullet; \mathcal{S}^1)$ by an S^1 -central extension as in Eq. (28) above. Without loss of generality, we may assume that $\Gamma' = \Gamma$. Let (U_i) be an open cover of Γ such that $\tilde{\Gamma} \rightarrow \Gamma$ admits a section over each U_i , denoted by $\varphi_i : U_i \rightarrow \tilde{\Gamma}$. Then the Čech 2-cocycle $c_{ijk} \in \check{C}^2(\mathcal{U}, \mathcal{S}^1)$ associated to the central extension is defined by the equation

$$\varphi_i(g)\varphi_j(h) = \varphi_k(gh)c_{ijk}(g, h) \quad (29)$$

($g \in U_i$, $h \in U_j$, $gh \in U_k$). Therefore, for all $x \in S\Gamma$ and $g \in \Gamma$ such that $t(g) = s(x)$, $x \in U_i$, $g \in U_j$, $xg \in U_k$, $x^g \in U_l$, we have

$$\varphi_i(x)\varphi_j(g) = \varphi_k(xg)c_{ijk}(x, g) \quad (30)$$

$$\varphi_j(g)\varphi_l(x^g) = \varphi_k(xg)c_{jlk}(g, x^g). \quad (31)$$

Comparing Eq. (30) and Eq. (31), we get

$$\varphi_j(g)\varphi_l(x^g)\varphi_j(g)^{-1} = \varphi_i(x)\frac{c_{jlk}(g, x^g)}{c_{ijk}(x, g)}. \quad (32)$$

The quantity $\frac{c_{jlk}(g, x^g)}{c_{ijk}(x, g)}$ thus does not depend on j and k . Let us denote it by $\psi_{il}(x, g)$. Then Eq. (32) reads

$$\gamma \cdot \varphi_l(s(\gamma)) = \varphi_i(t(\gamma))\psi_{il}(\gamma)$$

for all $\gamma \in (\Lambda\Gamma)_{U_i}^{U_i}$, where $(\gamma, p) \mapsto \gamma \cdot p$ denotes the action of Γ on the principal bundle $S\tilde{\Gamma}$. This shows that $\tau[c] = [\psi]$.

On the other hand, $(T_1 c)_{il}(x, g) = \frac{c_{ijk}(x, g)}{c_{jkl}(g, x^g)} = \psi_{il}(x, g)^{-1}$. Hence $T_1 = -\tau$. \square

3.3 Multiplicative cochains

Following the notations of Section 3.2, we assume that $N \xrightarrow{\varphi} \Gamma$ is a crossed module.

Definition 3.11 Let \mathcal{A} be an abelian sheaf over the bi-simplicial space $N_\bullet \rtimes \Gamma_\bullet$. A cochain $c \in \check{C}^n((N_k \rtimes \Gamma)_\bullet; \mathcal{A})$ is said to be r -multiplicative ($r \geq 1$) if $\partial' c = \partial b$ for some $(r-1)$ -multiplicative $b \in \check{C}^{n-1}((N_{k+1} \rtimes \Gamma)_\bullet; \mathcal{A})$. For $r = 0$, every cochain is said to be 0-multiplicative.

In other words, c is r -multiplicative if $\partial' c$ is a coboundary for the total complex $(\check{C}^p((N_{k+q} \rtimes \Gamma)_\bullet; \mathcal{A}), \partial, \partial')_{p \geq 0, 1 \leq q \leq r}$.

From now on, we assume that \mathcal{A} is a sheaf over the Δ_2 -space $(N \rtimes \Gamma)_{\bullet\bullet}$, and therefore it is also a sheaf over the bi-simplicial space $N_\bullet \rtimes \Gamma_\bullet$. The following lemma shows that the maps T_k enable us to produce many multiplicative cochains.

Lemma 3.12 Assume that $\omega \in \check{C}^n(\Gamma_\bullet; \mathcal{A})$ satisfies $\partial\omega = 0$. Then, for all $k \geq 1$, $T_k\omega \in \check{C}^{n-k}((N_k \rtimes \Gamma)_\bullet; \mathcal{A})$ is ∞ -multiplicative.

PROOF. We show, by induction on r , that $T_k\omega$ is r -multiplicative for all k and all ω as in the lemma. For $r = 0$, this is obvious. Let $r \geq 1$, and assume that the assertion is true up to $(r-1)$. According to Lemma 3.2, $\partial' T_k\omega = T_{k+1}\partial\omega + (-1)^k \partial T_{k+1}\omega = \pm \partial T_{k+1}\omega$. Since $T_{k+1}\omega$ is $(r-1)$ -multiplicative by the induction assumption, it follows that $T_k\omega$ is r -multiplicative. \square

Remark 3.13 It is easy to check that c is r -multiplicative if and only if $c + \partial c'$ is r -multiplicative. Hence one can talk about r -multiplicative cohomology classes. Therefore, any element in the image of $T_1 : \check{H}^n(\Gamma_\bullet; \mathcal{A}) \rightarrow \check{H}^{n-1}((N \rtimes \Gamma)_\bullet; \mathcal{A})$ is ∞ -multiplicative. In particular, for any groupoid Γ , any element in the image of the transgression map of $T_1 : \check{H}^n(\Gamma_\bullet; \mathcal{A}) \rightarrow \check{H}^{n-1}(\Lambda\Gamma_\bullet; \mathcal{A})$ is ∞ -multiplicative.

For $\mathcal{A} = \mathcal{S}^1$ and $n = 3$, we are led to the following

Corollary 3.14 *Let $N \xrightarrow{\varphi} \Gamma$ be a crossed module. For any $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$, let $c = T_1 e \in \check{C}^2((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$, $b = -T_2 e \in \check{C}^1((N_2 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ and $a = -T_3 e \in \check{C}^0((N_3 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$. Then we have*

$$\partial c = 0, \quad \partial' c = \partial b, \quad \text{and } \partial' b = \partial a. \quad (33)$$

Such a triple (c, b, a) is called a *multiplicator*.

Definition 3.15 For a crossed module $N \xrightarrow{\varphi} \Gamma$, a *multiplicator* is a triple (c, b, a) , where $c \in \check{C}^2((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$, $b \in \check{C}^1((N_2 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ and $a \in \check{C}^0((N_3 \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ such that Eq. (33) holds.

Remark 3.16 Multiplicators are closely related to “multiplicative bundle gerbes” studied in [14, 16]. They are also closely related to “simplicial gerbes” of Brylinski [10], which was later further developed by Stevenson under the name “simplicial bundle gerbes” [38].

3.4 Compatibility of the maps T_1 in Čech and de Rham cohomology

Let $N \xrightarrow{\varphi} \Gamma$ be a crossed module, and $N \rightrightarrows N_0$ and $\Gamma \rightrightarrows \Gamma_0$ are proper Lie groupoids. The purpose of this subsection is to compare the two transgression maps $T_1 : \check{H}^{i+1}(\Gamma_\bullet; \mathcal{S}^1) \rightarrow \check{H}^i((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ and $T_1 : H_{dR}^{i+1}(\Gamma_\bullet) \rightarrow H_{dR}^i((N \rtimes \Gamma)_\bullet)$. First we state two general results.

Lemma 3.17 *Let $0 \rightarrow \mathcal{I} \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow 0$ be an exact sequence of abelian sheaves over a Δ_2 -space $M_{\bullet\bullet}$. Then the diagram*

$$\begin{array}{ccc} H^i(M_{0,\bullet}; \mathcal{B}) & \xrightarrow{-T_1} & H^{i-1}(M_{1,\bullet}; \mathcal{B}) \\ \downarrow \lambda & & \downarrow \lambda \\ H^{i+1}(M_{0,\bullet}; \mathcal{I}) & \xrightarrow{T_1} & H^i(M_{1,\bullet}; \mathcal{I}) \end{array}$$

commutes, where λ denotes the boundary maps. A similar assertion holds for Čech cohomology as well.

PROOF. Follows immediately from the definition of the boundary maps and the relation (21). \square

Lemma 3.18 *Given an abelian sheaf \mathcal{A} over a Δ_2 -space $M_{\bullet\bullet}$, the following diagram*

$$\begin{array}{ccc} H^i(M_{0,\bullet}; \mathcal{A}) & \xrightarrow{T_1} & H^{i-1}(M_{1,\bullet}; \mathcal{A}) \\ \cong \downarrow & & \downarrow \cong \\ \check{H}^i(M_{0,\bullet}; \mathcal{A}) & \xrightarrow{T_1} & \check{H}^{i-1}(M_{1,\bullet}; \mathcal{A}) \end{array}$$

commutes.

PROOF. For any sheaf \mathcal{A} on a space X , denote by $\hat{\mathcal{A}}$ the sheaf such that $\hat{\mathcal{A}}(U) = \prod_{x \in U} \mathcal{A}_x$, i.e. a section of $\hat{\mathcal{A}}$ over U consists of a (generally discontinuous) map $x \mapsto f(x) \in \mathcal{A}_x$. As in [42, Lemma 7.2], let $I(\mathcal{A})^{k,l} = \tilde{\varepsilon}_{0*} \hat{\mathcal{A}}^{k,l+1}$ be the image sheaf [29] of $\hat{\mathcal{A}}^{k,l+1}$ under the

smooth map $\tilde{\varepsilon}_0 : M_{k,l+1} \rightarrow M_{k,l}$, where $\tilde{\varepsilon}_0$ is the map corresponding to $\varepsilon_0 : [l] \rightarrow [l+1]$, $i \mapsto i+1$, $\forall i$.

Define a Δ_2 -structure on $I(\mathcal{A})^\bullet$ as follows. For any $f = (a, b, c) \in \text{Hom}_{\Delta_2}((k, l), (k', l'))$, let $f' = (a', b', c') \in \text{Hom}_{\Delta_2}((k, l+1), (k', l'+1))$ be the unique element satisfying the conditions

$$\varepsilon_0 \circ f = f' \circ \varepsilon_0, \quad b'(0) = c'(0) = 0. \quad (34)$$

More precisely, $a' = a$, $b'(i+1) = b(i)$, $c'(0) = 0$, and $c'(i+1) = c(i) + 1$, $\forall i \geq 0$. For any $U \subset M_{k,l}$, $V \subset M_{k',l'}$ such that $f(V) \subset U$, from Eq. (34) it follows that $\tilde{f}'(\tilde{\varepsilon}_0^{-1}(V)) \subset \tilde{\varepsilon}_0^{-1}(U)$. Set

$$\tilde{f}_* = \tilde{f}'_* : I(\mathcal{A})^{k',l'}(V) (\cong \hat{\mathcal{A}}^{k',l'+1}(\tilde{\varepsilon}_0^{-1}(V))) \longrightarrow I(\mathcal{A})^{k,l}(U) (\cong \hat{\mathcal{A}}^{k,l+1}(\tilde{\varepsilon}_0^{-1}(U))).$$

It is simple to see that this indeed defines a Δ_2 -structure on $I(\mathcal{A})^\bullet$.

Consider the diagram

$$\begin{array}{ccc} H^i(M_{0,\bullet}; \mathcal{A}) & \xrightarrow{T_1} & H^{i-1}(M_{1,\bullet}; \mathcal{A}) \\ \lambda \uparrow & & \lambda \uparrow \\ H^{i-1}(M_{0,\bullet}; U(\mathcal{A})) & \xrightarrow{T_1} & H^{i-2}(M_{1,\bullet}; U(\mathcal{A})) \end{array}$$

where $U(\mathcal{A}) = \mathcal{A}/I(\mathcal{A})$ and the boundary maps λ come from the exact sequence of sheaves $0 \rightarrow I(\mathcal{A}) \rightarrow \mathcal{A} \rightarrow U(\mathcal{A}) \rightarrow 0$.

The diagram is anti-commutative, and the left vertical map is an isomorphism for $i \geq 2$ (see [42]). Similar assertions hold for Čech cohomology as well. Therefore, it suffices to prove the assertion for $i = 1$. The general conclusion follows from induction. Let us sketch an easy and direct argument below.

We have that $H^1(M_{0,\bullet}; \mathcal{A}) \cong \check{H}^1(M_{0,\bullet}; \mathcal{A})$, which consists of continuous sections $\gamma \mapsto \varphi(\gamma) \in \mathcal{A}_\gamma^{0,1}$ satisfying the equation $\tilde{\varepsilon}_1^* \varphi = \tilde{\varepsilon}_0^* \varphi + \tilde{\varepsilon}_2^* \varphi$. Also, $H^0(M_{1,\bullet}; \mathcal{A}) \cong \check{H}^0(M_{1,\bullet}; \mathcal{A})$, which is the space of Γ -invariant sections $\mathcal{A}^{1,0}(M_{1,0})^{inv} = \ker(\tilde{\varepsilon}_1^* - \tilde{\varepsilon}_0^*)$.

The maps $T_1 : H^1(M_{0,\bullet}; \mathcal{A}) \rightarrow H^0(M_{1,\bullet}; \mathcal{A})$ and $T_1 : \check{H}^1(M_{0,\bullet}; \mathcal{A}) \rightarrow \check{H}^0(M_{1,\bullet}; \mathcal{A})$ are both equal to $\tilde{\rho}^*$, where $\rho \in \text{Hom}_{\Delta_2}((0, 1), (1, 0))$ is the morphism $(\emptyset, \text{Id}, 0)$ (note that in the case of a crossed module $N \xrightarrow{\varphi} \Gamma$, $\tilde{\rho}$ coincides with the map $\varphi : N \rightarrow \Gamma$).

Let us explain why $T_1 = \tilde{\rho}^*$ for instance in sheaf cohomology. Represent an element of $H^1(M_{0,\bullet}; \mathcal{A})$ by a map $\gamma \mapsto \varphi(\gamma)$ as above. Let (\mathcal{A}_\bullet) be an injective resolution of \mathcal{A} . In the complex $\oplus_{p,q} \mathcal{A}_q(M_{0,p})$, the cohomology class $[\varphi]$ is represented by the cochain $\varphi \in \mathcal{A}(M_{0,1})$, where $\mathcal{A}(M_{0,1})$ is viewed as a subspace of $\mathcal{A}_0(M_{0,1}) \oplus \mathcal{A}_1(M_{0,0})$ using the injection $\mathcal{A} \rightarrow \mathcal{A}_0$. By definition of T_1 , we have $T_1(\varphi) = \tilde{f}_\sigma^* \varphi \in \mathcal{A}(M_{1,0})$ where $\sigma \in S_{1,0}$ is the unique $(1, 0)$ -shuffle. By definition of f_σ (see (12)), we have $f_\sigma = \rho$. \square

Proposition 3.19 *Let $N \xrightarrow{\varphi} \Gamma$ be a crossed module, and $N \rightrightarrows N_0$ and $\Gamma \rightrightarrows \Gamma_0$ proper Lie groupoids. Denote by λ the boundary maps in the induced long exact sequence of cohomology corresponding to the short exact sequence of sheaves*

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{R} \rightarrow \mathcal{S}^1 \rightarrow 0.$$

The following diagram

$$\begin{array}{ccc}
\check{H}^i(\Gamma_\bullet; \mathcal{S}^1) & \xrightarrow{-T_1} & \check{H}^{i-1}((N \rtimes \Gamma)_\bullet; \mathcal{S}^1) \\
\downarrow \lambda & & \downarrow \lambda \\
\check{H}^{i+1}(\Gamma_\bullet; \mathbb{Z}) & \xrightarrow{T_1} & \check{H}^i((N \rtimes \Gamma)_\bullet; \mathbb{Z}) \\
\downarrow \iota & & \downarrow \iota \\
\check{H}^{i+1}(\Gamma_\bullet; \mathbb{R}) & \xrightarrow{T_1} & \check{H}^i((N \rtimes \Gamma)_\bullet; \mathbb{R}) \\
\downarrow \psi & & \downarrow \psi \\
H_{dR}^{i+1}(\Gamma_\bullet) & \xrightarrow{T_1} & H_{dR}^i((N \rtimes \Gamma)_\bullet)
\end{array}$$

commutes. Here the maps ι are $\check{H}^*(-; \mathbb{Z}) \rightarrow \check{H}^*(-; \mathbb{Z}) \otimes \mathbb{R}$, and ψ are the natural isomorphisms. Moreover, if $i \geq 2$, the maps λ are isomorphisms.

PROOF. The fact that the first square commutes follows from Lemma 3.17. The boundary maps λ are isomorphisms because $\check{H}^i(\Gamma_\bullet; \mathcal{R}) \cong H^i(\Gamma_\bullet; \mathcal{R}) = 0$, for all $i \geq 1$, since Γ is proper [19]. Commutativity of the second square is obvious. Commutativity of the third square follows from Lemma 3.18, since the de Rham cohomology is isomorphic to the sheaf cohomology of \mathbb{R} . \square

Consider the special case that $N = \Gamma = G$ is a compact, connected and simply connected simple Lie group, with G acting on $N = G$ by conjugation. It is standard that

$$\check{H}^3(G_\bullet; \mathcal{S}^1) \cong H^3(G_\bullet; \mathcal{S}^1) = \mathbb{Z}$$

and

$$\check{H}^2((G \rtimes G)_\bullet; \mathcal{S}^1) \cong H^2((G \rtimes G)_\bullet; \mathbb{Z}) = \mathbb{Z}.$$

As a consequence of Proposition 3.19, we see that the transgression map $T_1 : \check{H}^3(G_\bullet; \mathcal{S}^1) \rightarrow \check{H}^2((G \rtimes G)_\bullet; \mathcal{S}^1)$ is indeed an isomorphism.

Corollary 3.20 *Let G be a compact, connected and simply connected simple Lie group. Then the transgression map $T_1 : \check{H}^3(G_\bullet; \mathcal{S}^1) \rightarrow \check{H}^2((G \rtimes G)_\bullet; \mathcal{S}^1)$ is an isomorphism.*

PROOF. According to Proposition 3.19, it suffices to show that $T_1 : H_{dR}^4(G_\bullet) \rightarrow H_{dR}^3((G \rtimes G)_\bullet)$ maps the canonical generator to the canonical generator. It is well known that the canonical generator of $H_{dR}^4(G_\bullet)$ is given by $\lambda - \Omega \in \Omega^2(G \times G) \oplus \Omega^3(G)$ [8, 45], where $\lambda = (\theta_1, \bar{\theta}_2)$, $\Omega = \frac{1}{6}(\theta, [\theta, \theta])$ and θ (resp. $\bar{\theta}$) denotes the left (resp. right) Maurer-Cartan form on G . Here (\cdot, \cdot) stands for the Killing form on \mathfrak{g} . According to Eq. (3), the image of $\lambda - \Omega$ by T_1 is $\omega - \Omega \in \Omega^2(G \times G) \oplus \Omega^3(G)$, where $\omega = \lambda - f^*\lambda$ and $f(x, g) = (g, x^g)$. A simple calculation shows that

$$\omega|_{(x, g)} = -[(g^*\bar{\theta}, Ad_x g^*\bar{\theta}) + (g^*\bar{\theta}, x^*(\theta + \bar{\theta}))], \quad (35)$$

where (x, g) denotes the coordinate in $G \times G$, and $x^*\theta$ and $g^*\theta$ are, respectively, the \mathfrak{g} -valued one-forms on $G \times G$ obtained by pulling back θ via the first and second projections, and similarly for $x^*\bar{\theta}$. According to [6], $\omega - \Omega$ is indeed the canonical generator of $H_{dR}^3((G \rtimes G)_\bullet)$. \square

⁵Note that the transformation groupoid in [6] uses the left conjugation action. The map $\phi : G \rtimes G \rightarrow G \rtimes G$ establishing the groupoid isomorphisms from our convention to that in [6] is given by $\phi(x, g) = (g, g^{-1}xg)$.

Remark 3.21 In [12], Brylinski and McLaughlin studied the transgression map from $H^4(BG)$ to $H^3(BLG)$, and the relation with Segal-Witten reciprocity law. It would be interesting to investigate the connection with our transgression map here.

4 Ring structure on the equivariant twisted K -theory

In this section, we assume that

- (i) $N \xrightarrow{\varphi} \Gamma$ is a crossed module, where $N \rightrightarrows N_0$ and $\Gamma \rightrightarrows \Gamma_0$ are proper Lie groupoids such that the source (or target) map $N \rightarrow N_0$ is Γ -equivariantly K -oriented (see definition below); and
- (ii) (c, b, a) is a multiplier, where $c \in \check{C}^2((N \rtimes \Gamma)_\bullet, \mathcal{S}^1)$, $b \in \check{C}^1((N_2 \rtimes \Gamma)_\bullet, \mathcal{S}^1)$ and $a \in \check{C}^0((N_3 \rtimes \Gamma)_\bullet, \mathcal{S}^1)$. That is,

$$\partial c = 0, \quad \partial' c = \partial b, \quad \text{and} \quad \partial' b = \partial a.$$

The main purpose of this section is to construct a ring structure on the twisted equivariant K -theory groups $K_{c,\Gamma}^*(N)$ for any given multiplier (c, b, a) , and investigate how the ring structure depends on the choice of these multipliers.

4.1 2-cocycles and S^1 -central extensions

Let us briefly recall how an \mathcal{S}^1 -valued 2-cocycle determines an S^1 -central extension. Given a groupoid $\Gamma \rightrightarrows \Gamma_0$, a cover $\mathcal{U} = (\mathcal{U}^k)_{k \in \mathbb{N}}$ of Γ_\bullet with $\mathcal{U}^k = (U_j^k)_{j \in I_k}$, and a 2-cocycle $c \in \check{Z}^2(\sigma\mathcal{U}; \mathcal{S}^1)$, c can be viewed as a family (c_i) , with $i = (i_0, i_1, i_2, i_{01}, i_{02}, i_{12}, i_{012}) \in I_0^3 \times I_1^3 \times I_2$. Here c_i is an S^1 -valued function on some open set of Γ_2 . The 2-cocycle c determines an S^1 -central extension

$$S^1 \rightarrow \tilde{\Gamma}^c \xrightarrow{\pi} \Gamma^c \rightrightarrows \Pi_{i \in I_0} U_i^0, \quad (36)$$

where $\Gamma^c \rightrightarrows \Pi_{i \in I_0} U_i^0$ is the pull-back groupoid $\Gamma[\mathcal{U}^0] \rightrightarrows \Pi_{i \in I_0} U_i^0$ given by $\Gamma[\mathcal{U}^0] = \{(i, j, \gamma) \mid i, j \in I_0 \text{ and } \gamma \in \Gamma_{U_j^0}^{U_i^0}\}$. The groupoid $\tilde{\Gamma}^c \rightrightarrows \Pi_{i \in I_0} U_i^0$ is

$$\tilde{\Gamma}^c = \{(k, i, j, \gamma, \lambda) \mid i, j \in I_0, k \in I_1, \gamma \in \Gamma_{U_j^0}^{U_i^0} \cap U_k^1, \lambda \in S^1\} / \sim, \quad (37)$$

where \sim is an appropriate equivalence relation, $\pi(k, i, j, \gamma, \lambda) = (i, j, \gamma)$, and the product on $\tilde{\Gamma}^c$ is defined by

$$(i_{01}, i_0, i_1, \gamma_1, \lambda)(i_{12}, i_1, i_2, \gamma_2, \mu) = (i_{02}, i_0, i_2, \gamma_1 \gamma_2, \lambda \mu c_i(\gamma_1, \gamma_2)). \quad (38)$$

We refer the reader to [42] for details.

Recall that $\check{C}^m(\Gamma_\bullet; \mathcal{S}^1) = \lim_{\mathcal{U}} \check{C}^m(\sigma\mathcal{U}; \mathcal{S}^1)$, where \mathcal{U} runs over all covers of the form $(U_\gamma^k)_{\gamma \in \Gamma_k}$, $\gamma \in U_\gamma^k$, and \mathcal{U}' is finer than \mathcal{U} if $U_\gamma'^k \subset U_\gamma^k$ for all k and all $\gamma \in \Gamma_k$.

Lemma 4.1 *For any open cover \mathcal{U} of Γ_\bullet , if $c^1, c^2 \in \check{Z}^2(\sigma\mathcal{U}; \mathcal{S}^1)$, and $b \in \check{C}^1(\sigma\mathcal{U}; \mathcal{S}^1)$ satisfy $c^1 - c^2 = \partial b$, then there is an isomorphism of S^1 -central extensions*

$$\Phi_{c^2, b, c^1}^{\mathcal{U}} : (\tilde{\Gamma}^{c_1} \xrightarrow{\pi} \Gamma^{c_1}) \longrightarrow (\tilde{\Gamma}^{c_2} \xrightarrow{\pi} \Gamma^{c_2})$$

satisfying the following properties:

- (a) $\Phi_{c^3, b', c^2}^{\mathcal{U}} \Phi_{c^2, b, c^1}^{\mathcal{U}} = \Phi_{c^3, b+b', c^1}^{\mathcal{U}}$, where $c^1 - c^2 = \partial b$ and $c^2 - c^3 = \partial b'$;
- (b) if $b = \partial a$ for some $a \in \check{C}^0(\sigma\mathcal{U}; \mathcal{S}^1)$, then $\Phi_{c, b, c}^{\mathcal{U}} : \tilde{\Gamma}^c \rightarrow \tilde{\Gamma}^c$ is the multiplier by $a(t(g))a(s(g))^{-1}$, $\forall g \in \tilde{\Gamma}^c$; and
- (c) if \mathcal{U}' is a refinement of \mathcal{U} and c'^1, c'^2 and b' are the images of c^1, c^2 and b under the canonical map $\check{C}^*(\sigma\mathcal{U}; \mathcal{S}^1) \rightarrow \check{C}^*(\sigma\mathcal{U}'; \mathcal{S}^1)$, then the following diagram

$$\begin{array}{ccc} \tilde{\Gamma}^{c'^1} & \longrightarrow & \tilde{\Gamma}^{c^1} \\ \Phi_{c'^2, b', c'^1}^{\mathcal{U}'} \downarrow & & \downarrow \Phi_{c^2, b, c^1}^{\mathcal{U}} \\ \tilde{\Gamma}^{c'^2} & \longrightarrow & \tilde{\Gamma}^{c^2} \end{array}$$

commutes, where the horizontal maps are Morita morphisms of S^1 -central extensions.

PROOF. In terms of notation (37), we denote by φ_k^1 and φ_k^2 the maps

$$\varphi_k^1(i, j, \gamma) = [(k, i, j, \gamma, 1)] \in \tilde{\Gamma}^{c^1}, \quad \varphi_k^2(i, j, \gamma) = [(k, i, j, \gamma, 1)] \in \tilde{\Gamma}^{c^2}.$$

Then φ_k^1 and φ_k^2 are local sections of the S^1 -principal bundles $\tilde{\Gamma}^{c^1} \xrightarrow{\pi} \Gamma^{c^1}$ and $\tilde{\Gamma}^{c^2} \xrightarrow{\pi} \Gamma^{c^2}$, respectively. The relation $c^2 - c^1 = \partial b$ means that

$$c_i^2(\gamma_1, \gamma_2) c_i^1(\gamma_1, \gamma_2)^{-1} = b_{i_0 i_1 i_{01}}(\gamma_1) b_{i_1 i_2 i_{12}}(\gamma_2) b_{i_0 i_2 i_{02}}(\gamma_1 \gamma_2)^{-1}, \quad (39)$$

$$\forall i = (i_0, i_1, i_2, i_{01}, i_{02}, i_{12}, i_{012}) \in I_0^3 \times I_1^3 \times I_2. \quad (40)$$

By applying Eq. (38) to the 2-cocycles c^1 and c^2 , we are led to

$$\varphi_{i_{01}}^j(i_0, i_1, \gamma_1) \varphi_{i_{12}}^j(i_1, i_2, \gamma_2) = \varphi_{i_{02}}^j(i_0, i_2, \gamma_1 \gamma_2) c_i^j(\gamma_1, \gamma_2), \quad j = 1, 2. \quad (41)$$

Define a map $\Phi_{c^2, b, c^1}^{\mathcal{U}} : \tilde{\Gamma}^{c^1} \rightarrow \tilde{\Gamma}^{c^2}$ by

$$[(k, i, j, \gamma, \lambda)] \longrightarrow [(k, i, j, \gamma, \lambda b_{ijk}(\gamma)^{-1})]. \quad (42)$$

That is,

$$\lambda \varphi_k^1(i, j, \gamma) \mapsto \lambda \varphi_k^2(i, j, \gamma) b_{ijk}(\gamma)^{-1} \quad (43)$$

for all $\lambda \in S^1$ and $\gamma \in U_k^1 \cap \Gamma_{U_j^0}^{U_i^0}$.

Let us check that $\Phi_{c^2, b, c^1}^{\mathcal{U}}$ is well-defined, i.e. it is independent of the choice of k . To simplify notations, after replacing Γ by $\Gamma[\mathcal{U}_0]$ and the cocycles c^j by $c_{i_{01} i_{12} i_{02}}^j((i_0, i_1, \gamma_1), (i_1, i_2, \gamma_2)) = c_{i_0 i_1 i_2 i_{01} i_{12} i_{02}}^j(\gamma_1, \gamma_2)$, we can assume that the cover \mathcal{U}^0 consists of the entire open set Γ_0 . Thus Eq. (43) reads

$$\lambda \varphi_k^1(\gamma) \mapsto \lambda \varphi_k^2(\gamma) b_k(\gamma)^{-1}. \quad (44)$$

Using Eq. (41), we get $\varphi_m^j(s(\gamma)) \varphi_m^j(s(\gamma)) = \varphi_m^j(s(\gamma)) c_{mmm}^j(s(\gamma), s(\gamma))$. Thus one can identify $\varphi_m^j(s(\gamma))$ with the complex number $c_{mmm}^j(s(\gamma), s(\gamma))$. By Eq. (41) again, we obtain $\varphi_l^j(\gamma) \varphi_m^j(s(\gamma)) = \varphi_k^j(\gamma) c_{lmk}^j(\gamma, s(\gamma))$. For $j = 1$, after replacing $\varphi_m^j(s(\gamma))$ by $c_{mmm}^j(s(\gamma), s(\gamma))$, we have

$$\varphi_l^1(\gamma) = \lambda_1 \varphi_k^1(\gamma) \quad (45)$$

with $\lambda_j = c_{lmk}^j(\gamma, s(\gamma))/c_{mmm}^j(s(\gamma), s(\gamma))$ ($j = 1, 2$). From Eq. (39), we obtain that $\lambda_2 = \lambda_1 b_l(\gamma) b_k(\gamma)^{-1}$. Therefore

$$\varphi_l^2(\gamma) b_l(\gamma)^{-1} = \lambda_1 \varphi_k^2(\gamma) b_k(\gamma)^{-1}. \quad (46)$$

Comparing Eqs. (44)-(46), we see that $\Phi_{c^2, b, c^1}^{\mathcal{U}}$ is indeed well-defined.

The fact that $\Phi_{c^2, b, c^1}^{\mathcal{U}}$ is a groupoid morphism now immediately follows from Eqs. (39)-(41). Relation (a) is also clear from Eq. (43).

Let us prove (b). Again, for simplicity, we may assume that \mathcal{U}^0 is a cover consisting of just one open set Γ_0 . Hence, by assumption, $b_i(g) = a(t(g))a(s(g))^{-1}$ for all $i \in I_1$ and $g \in U_i^1$. Hence the automorphism of the central extension $S^1 \rightarrow \Gamma^c \rightarrow \Gamma^c$ given by Eq. (42) is

$$\Phi_{c, b, c}^{\mathcal{U}}(\gamma) = \gamma a(s(\gamma)) a(t(\gamma))^{-1}, \quad \forall \gamma \in \tilde{\Gamma}^c$$

Let us show (c). By definition, a refinement is a map $\psi : I'_n \rightarrow I_n$ ($n \in \mathbb{N}$) such that $U_i^m \subset U_{\psi(i)}^n$. Consider the following diagram

$$\begin{array}{ccc} \tilde{\Gamma}^{c'^1} & \longrightarrow & \tilde{\Gamma}^{c^1} \\ \Phi_{c'^2, b', c'^1}^{\mathcal{U}} \downarrow & & \downarrow \Phi_{c^2, b, c^1}^{\mathcal{U}} \\ \tilde{\Gamma}^{c'^2} & \longrightarrow & \tilde{\Gamma}^{c^2} \end{array}$$

Here the horizontal maps are defined by $[k, i, j, \gamma, \lambda] \mapsto [\psi(k), \psi(i), \psi(j), \gamma, \lambda]$, which are clearly Morita morphisms of S^1 -central extensions. It is immediate to check that the above diagram indeed commutes. \square

4.2 The C^* -algebra associated to a 2-cocycle

Let us recall the construction of the C^* -algebra associated to an S^1 -central extension

$$S^1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightrightarrows \Gamma_0,$$

where $\Gamma \rightrightarrows \Gamma_0$ is a Lie groupoid, or, more generally, a locally compact groupoid endowed with a Haar system $(\lambda^x)_{x \in \Gamma_0}$. Let $L = \tilde{\Gamma} \times_{S^1} \mathbb{C}$ be the complex line bundle associated to the S^1 -principal bundle $\tilde{\Gamma} \rightarrow \Gamma$. Then $L \rightarrow \Gamma \rightrightarrows M$ is equipped with an associative bilinear product

$$\begin{aligned} L_g \otimes L_h &\rightarrow L_{gh} \quad \forall (g, h) \in \Gamma_2 \\ (\xi, \eta) &\mapsto \xi \cdot \eta \end{aligned}$$

and an antilinear involution

$$\begin{aligned} L_g &\rightarrow L_{g^{-1}} \\ \xi &\mapsto \xi^* \end{aligned}$$

satisfying the following properties:

- the restriction of the line bundle to the unit space M is isomorphic to the trivial line bundle $M \times \mathbb{C} \rightarrow M$;

- $\forall \xi, \eta \in L_g, \langle \xi, \eta \rangle = \xi^* \cdot \eta \in L_{s(g)} \cong \mathbb{C}$ defines a scalar product;
- $(\xi \cdot \eta)^* = \eta^* \cdot \xi^*$.

The space $C_c(\Gamma, L)$ of continuous, compactly supported sections of the line bundle $L \rightarrow \Gamma$ is endowed with a convolution product:

$$(\xi * \eta)(g) = \int_{h \in \Gamma^{t(g)}} \xi(h) \cdot \eta(h^{-1}g) d\lambda^{t(g)}(h),$$

and the adjoint

$$\xi^*(g) = (\xi(g^{-1}))^*,$$

where $\xi(h) \cdot \eta(h^{-1}g)$ is understood as the product $L_h \otimes L_{h^{-1}g} \rightarrow L_g$.

For all $x \in \Gamma_0$, let \mathcal{H}_x be the Hilbert space obtained by completing $C_c(\Gamma, L)$ with respect to the scalar product

$$\langle \xi, \eta \rangle = (\xi^* * \eta)(x) = \int_{g \in \Gamma^x} \langle \xi(g^{-1}), \eta(g^{-1}) \rangle d\lambda^x(g).$$

Let $(\pi_x(f))\xi = f * \xi, \forall f \in C_c(\Gamma, L), \xi \in \mathcal{H}_x$. Then $f \mapsto \pi_x(f)$ is a $*$ -representation of $C_c(\Gamma, L)$ in \mathcal{H}_x . The C^* -algebra $C_r^*(\Gamma, L)$ is, by definition, the completion of $C_c(\Gamma, L)$ with respect to the norm: $\sup_{x \in \Gamma_0} \|\pi_x(f)\|$.

Let $\Gamma \rightrightarrows \Gamma_0$ be a locally compact groupoid with Haar system. Let $c \in \check{Z}^2(\sigma\mathcal{U}; S^1)$ be a Čech 2-cocycle on an open cover \mathcal{U} of Γ . Then c determines an S^1 -central extension as given by Eq. (36), where Γ^c is Morita equivariant to Γ . Denote by $L^c = \tilde{\Gamma}^c \times_{S^1} \mathbb{C}$ the associated complex line bundle. By $C_r^*(\Gamma, c)$, we denote the C^* -algebra $C_r^*(\Gamma^c, L^c)$ and call the C^* -algebra associated to the 2-cocycle c .

4.3 S^1 -equivariant gerbes

The main result of this subsection is the following

Theorem 4.2 *Assume that $\Gamma \rightrightarrows \Gamma_0$ is a Lie groupoid acting on a manifold N via $J : N \rightarrow \Gamma_0$. Let \mathcal{U} be a cover of $(N \rtimes \Gamma)_\bullet$. Any Čech 2-cocycle $c \in \check{Z}^2(\mathcal{U}, S^1)$ canonically determines an S^1 -central extension of the form $\tilde{H} \rtimes \Gamma \rightarrow H \rtimes \Gamma \rightrightarrows M$, where $\tilde{H} \rightarrow H \rightrightarrows M$ is a Γ -equivariant S^1 -central extension and $H \rightrightarrows M$ is Morita equivalent to $N \rightrightarrows N$, such that the class of this central extension is equal to $[c] \in \check{H}^2((N \rtimes \Gamma)_\bullet; S^1)$.*

We first prove the following special case.

Proposition 4.3 *Let $\Gamma \rightrightarrows \Gamma_0$ be a Lie groupoid, and \mathcal{U} a cover of Γ_\bullet . Any Čech 2-cocycle $c \in \check{Z}^2(\mathcal{U}, S^1)$ canonically determines an S^1 -central extension of the form $\tilde{H} \rtimes \Gamma \rightarrow H \rtimes \Gamma \rightrightarrows M$, where $\tilde{H} \rightarrow H \rightrightarrows M$ is a Γ -equivariant S^1 -central extension and $H \rightrightarrows M$ is Morita equivalent to $\Gamma_0 \rightrightarrows \Gamma_0$, such that the class of this central extension is equal to $[c] \in \check{H}^2(\Gamma_\bullet; S^1)$.*

Lemma 4.4 *Assume that M is a right Γ -space via the momentum map $J : M \rightarrow \Gamma_0$, which is a surjective submersion. Then the pull-back groupoid $\Gamma[M] \rightrightarrows M$ via J is naturally isomorphic to the crossed-product $H \rtimes \Gamma \rightrightarrows M$, where $H \rightrightarrows M$ is the equivalence groupoid $M \times_{\Gamma_0} M \rightrightarrows M$, and Γ acts on $H \rightrightarrows M$ by natural automorphisms.*

PROOF. By definition, $\Gamma[M] = \{(x, y, g) | J(x) = t(g), J(y) = s(g)\}$. One checks directly that $\chi : H \rtimes \Gamma \rightarrow \Gamma[M]$ given by $(x, y, g) \mapsto (x, yg^{-1}, g)$ is indeed a groupoid isomorphism. \square

Proof of Proposition 4.3

PROOF. From Section 4.1, we know that c canonically determines

(a) a surjective submersion $\pi : U \rightarrow \Gamma_0$; and

(b) an S^1 -central extension

$$S^1 \rightarrow \tilde{\Gamma} \xrightarrow{p} \Gamma[U] \rightrightarrows U, \quad (47)$$

where $U = \coprod_{i \in I_0} U_i^0$. Let $M = U \times_{\Gamma_0} \Gamma$. It is clear that Γ acts on M by $(x, g) \cdot \gamma = (x, g\gamma)$ with the momentum map $\sigma : M \rightarrow \Gamma_0$ given by $\sigma(x, g) = s(g)$, $\forall (x, g) \in U \times_{\Gamma_0} \Gamma$. Thus according to Lemma 4.4, the pull back groupoid $\sigma^*\Gamma$ is isomorphic to the crossed-product $H \rtimes \Gamma \rightrightarrows M$, where $H = M \times_{\sigma, \Gamma_0, \sigma} M = \{(x', y') \in M \times M | \sigma(x') = \sigma(y')\}$ and Γ acts on H by natural automorphisms. Now consider the map $\tau : M \rightarrow \Gamma_0$ given by $\tau(x, g) = t(g)$, $\forall (x, g) \in U \times_{\Gamma_0} \Gamma$. It is clear that τ is also a surjective submersion. Introduce a map

$$\rho : \sigma^*\Gamma \longrightarrow \tau^*\Gamma \quad (48)$$

$$((x, g), (y, h), r) \longrightarrow ((x, g), (y, h), grh^{-1}). \quad (49)$$

It is simple to check that ρ is a groupoid isomorphism. It thus follows that the groupoids $H \rtimes \Gamma \rightrightarrows M$ and $\tau^*\Gamma \rightrightarrows M$ are isomorphic. The isomorphism $\psi : H \rtimes \Gamma \rightarrow \tau^*\Gamma$ is given by $\psi((x, g), (y, h), \gamma) = ((x, g), (y, h\gamma), gh^{-1})$, which is the composition of the isomorphism χ in the proof of Lemma 4.4 with the isomorphism ρ as in Eq. (49). On the other hand, since $\tau = \pi \circ p_1$, where $p_1 : M \rightarrow U$ is the natural projection, then $\tau^*\Gamma \xrightarrow{p_1} \pi^*\Gamma (= \Gamma[U])$ is a Morita morphism of groupoids. Here, by abuse of notations, we use the same symbol p_1 to denote the groupoid morphism on the pull-back groupoids induced by $p_1 : M \rightarrow U$. Therefore, it follows that

$$\begin{array}{ccc} H \rtimes \Gamma & \xrightarrow{\varphi} & \Gamma[U] \\ \Downarrow & & \Downarrow \\ M & \xrightarrow{p_1} & U \end{array}$$

is a Morita morphism of groupoids, where $\varphi = p_1 \circ \psi$ can be expressed explicitly by $\varphi((x, g), (y, h), \gamma) = (x, y, gh^{-1})$. Let

$$\tilde{\Gamma}' = \varphi^*\tilde{\Gamma} = (H \rtimes \Gamma) \times_{\Gamma[U]} \tilde{\Gamma}.$$

Then it is clear that

$$S^1 \rightarrow \tilde{\Gamma}' \rightarrow H \rtimes \Gamma \rightrightarrows M \quad (50)$$

is an S^1 -central extension Morita equivalent to (47), and therefore it defines the same cohomology class $[c]$.

Identify H with the subgroupoid H' of $H \rtimes \Gamma$ via the embedding $i : h \mapsto (h, J_1(h))$, where $J_1 : H \rightarrow \Gamma_0$ is the momentum map of the Γ -action. Note that Γ naturally acts on H' by $(h, J_1(h))^r = (h^r, s(r))$, which coincides with the natural Γ -action on H under the identification $H' \xrightarrow{\sim} H$. Let $j = \varphi \circ i : H \rightarrow \Gamma[U]$. Now let

$$\tilde{H} = j^*\tilde{\Gamma} = H' \times_{\Gamma[U]} \tilde{\Gamma}$$

be the pull back S^1 -bundle, via j , of $\tilde{\Gamma} \xrightarrow{p} \Gamma[U]$ as given in Eq. (47).

The Γ -action on H' induces a Γ -action on \tilde{H} . Indeed, \tilde{H} can be naturally identified with $\{(h, \tilde{\gamma}) \in H \times \tilde{\Gamma} \mid j(h) = p(\tilde{\gamma})\}$, under which the Γ -action is given by $(h, \tilde{\gamma})^\gamma = (h^\gamma, \tilde{\gamma})$. This action is well-defined since j is Γ -invariant. It is clear from the construction that

$$S^1 \rightarrow \tilde{H} \xrightarrow{q} H \rightrightarrows M \quad (51)$$

is a Γ -equivariant S^1 -central extension. Therefore one obtains an S^1 -central extension

$$S^1 \rightarrow \tilde{H} \rtimes \Gamma \xrightarrow{q} H \rtimes \Gamma \rightrightarrows M, \quad (52)$$

which is clearly isomorphic to (50). Indeed the map $f : \tilde{H} \rtimes \Gamma \rightarrow \tilde{\Gamma}$, defined by $f(h, \tilde{\gamma}, \gamma) = \tilde{\gamma}$, is an S^1 -equivariant groupoid morphism and the diagram

$$\begin{array}{ccc} \tilde{H} \rtimes \Gamma & \xrightarrow{f} & \tilde{\Gamma} \\ \downarrow q & & \downarrow p \\ H \rtimes \Gamma & \xrightarrow{\varphi} & \Gamma[U] \end{array}$$

commutes. \square

Theorem 4.2 thus follows from Proposition 4.3 combining with the following

Lemma 4.5 *Assume that $\Gamma \rightrightarrows \Gamma_0$ is a Lie groupoid acting on a manifold N via $J : N \rightarrow \Gamma_0$. If a groupoid $H \rightrightarrows M$ admits an action of $N \rtimes \Gamma$ by automorphisms, then it must admit a Γ -action by automorphisms. Moreover, the crossed product groupoids $H \rtimes (N \rtimes \Gamma)$ and $H \rtimes \Gamma$ are canonically isomorphic.*

PROOF. This follows from a direct verification, which we will leave to the reader. \square

Corollary 4.6 *Under the same hypothesis as in Theorem 4.2, c canonically determines a Γ - C^* -algebra A_c and a Morita equivalence ν_c from the C^* -algebra $C_r^*(N \rtimes \Gamma, c)$ of the S^1 -central extension (47) to the crossed product $A_c \rtimes_r \Gamma$.*

Moreover, under the same assumptions as in Lemma 4.1, there is a Γ -equivariant Morita equivalence α_{c^2, b, c^1} from A_{c^1} to A_{c^2} such that the diagram

$$\begin{array}{ccc} C_r^*(\Gamma, c^1) & \xrightarrow{\nu_{c^1}} & A_{c^1} \rtimes_r \Gamma \\ \downarrow \Phi_{c^2, b, c^1}^{\mathcal{U}} & & \downarrow \alpha_{c^2, b, c^1} \rtimes \Gamma \\ C_r^*(\Gamma, c^2) & \xrightarrow{\nu_{c^2}} & A_{c^2} \rtimes_r \Gamma \end{array}$$

commutes. Moreover, the above diagram is compatible with refinements of covers as in Lemma 4.1.

PROOF. After replacing Γ by $N \rtimes \Gamma$, we may assume that $N = \Gamma_0$. Recall from the proof of Proposition 4.3 that the cover \mathcal{U} on which c is defined determines an étale map $\pi_c : U^c \rightarrow \Gamma_0$, a groupoid H^c , an action of Γ on H^c by automorphisms, and the map j_c which is the composition $H^c \xrightarrow{i_c} H^c \rtimes \Gamma \xrightarrow{\varphi_c} \Gamma[U^c]$.

The cocycle c determines an S^1 -central extension $S^1 \rightarrow \tilde{\Gamma}^c \xrightarrow{p_c} \Gamma[U^c]$, whose C^* -algebra is $C_r^*(\Gamma, c)$. The pull back of this S^1 -bundle $S^1 \rightarrow \tilde{H}^c \rightarrow H^c$ by the map j_c is a Γ -equivariant extension as shown in Eq. (51). We take A_c to be its C^* -algebra, which is thus endowed with an action of Γ .

Since there is an S^1 -equivariant isomorphism $f_c : \tilde{H}^c \rtimes \Gamma \rightarrow \varphi_c^* \tilde{\Gamma}^c$ (see the proof of Proposition 4.3), we obtain a canonical Morita equivalence ν_c . Since j_c depends only on \mathcal{U} , the isomorphism $\Phi_{c^2, b, c^1}^{\mathcal{U}} : (S^1 \rightarrow \tilde{\Gamma}^{c^1} \rightarrow \Gamma[U^{c^1}]) \rightarrow (S^1 \rightarrow \tilde{\Gamma}^{c^2} \rightarrow \Gamma[U^{c^2}])$ determines a Γ -equivariant isomorphism $\alpha_{c^2, b, c^1} : (S^1 \rightarrow \tilde{H}^{c^1} \rightarrow H^{c^1}) \rightarrow (S^1 \rightarrow \tilde{H}^{c^2} \rightarrow H^{c^2})$, and therefore an isomorphism $\alpha_{c^2, b, c^1} \rtimes \Gamma : (S^1 \rightarrow \tilde{H}^{c^1} \rtimes \Gamma \rightarrow H^{c^1} \rtimes \Gamma) \rightarrow (S^1 \rightarrow \tilde{H}^{c^2} \rtimes \Gamma \rightarrow H^{c^2} \rtimes \Gamma)$. It is easy to check that this isomorphism coincides with $\varphi_c^* \Phi_{c^2, b, c^1}^{\mathcal{U}}$, via the identification $f_c : \tilde{H}^c \rtimes \Gamma \rightarrow \varphi_c^* \tilde{\Gamma}^c$. Hence the commutative diagram follows. The last assertion is easy to check. \square

Remark 4.7 Let Γ be a Lie groupoid. Recall [35, 43] that the Brauer group $Br(\Gamma)$ of Γ is defined as the group of Morita equivalence classes of Γ -equivariant locally trivial bundles of C^* -algebras over Γ_0 whose fibers are the algebra of compact operators on a Hilbert space. It is known that $Br(\Gamma)$ is isomorphic to $H^2(\Gamma; \mathcal{S}^1)$ [42]. The corollary above enables us to realize such a bundle geometrically for a given Čech 2-cocycle $c \in \check{Z}^2(\Gamma; S^1)$.

We will need the following functorial property of the correspondence $c \mapsto A_c$:

Proposition 4.8 *Let Γ be a Lie groupoid acting on manifolds N' and N , and $f : N' \rightarrow N$ a Γ -equivariant map. Assume that $c \in \check{Z}^2(\mathcal{U}, S^1)$, where \mathcal{U} is a cover of $(N \rtimes \Gamma)_\bullet$. Then A_{f^*c} is canonically isomorphic to $f^*A_c \cong C_0(N') \otimes_{C_0(N)} A_c$. Moreover, this isomorphism is compatible with refinements of covers, and with Φ_b and α_b as in Corollary 4.6.*

(Note that A_c can be considered as a field of C^* -algebras over N , and hence f^*A_c is a field of C^* -algebras over N' .)

PROOF. We use the same notations as in the proof of Proposition 4.3. The groupoid H is a fibration over N , with each fiber $H_x = \{((i, \gamma), (j, \gamma')) \in I_0 \times \Gamma^x \times I_0 \times \Gamma^x \mid s(\gamma) \in U_i^0, s(\gamma') = U_j^0\}$. Similarly, the groupoid H' is a fibration over N' , and one immediately sees that the fiber H'_x over $x \in N'$ is isomorphic to $H_{f(x)}$. More precisely, there is a cartesian diagram

$$\begin{array}{ccc} H' & \longrightarrow & H \\ \downarrow & & \downarrow \\ N' & \xrightarrow{f} & N. \end{array}$$

Recall also from the proof of Proposition 4.3 that the central extension $S^1 \rightarrow \tilde{H} \rightarrow H$ is obtained by the pull back of the central extension of $\tilde{\Gamma} \rightarrow \Gamma[U]$ by the groupoid morphism $\varphi : H \rightarrow \Gamma[U]$. The central extension $S^1 \rightarrow \tilde{H}' \rightarrow H'$ admits a similar description. It is simple to see that the diagram

$$\begin{array}{ccc} H' & \longrightarrow & H \\ \downarrow \varphi' & & \downarrow \varphi \\ f^*\Gamma[U] & \longrightarrow & \Gamma[U] \end{array}$$

commutes. It thus follows that there exists an S^1 -equivariant map $\tilde{H}' \rightarrow \tilde{H}$ such that

$$\begin{array}{ccc} \tilde{H}' & \longrightarrow & \tilde{H} \\ \downarrow & & \downarrow \\ N' & \xrightarrow{f} & N \end{array}$$

commutes. In other words, considering $\tilde{H} \rightarrow N$ as an S^1 -central extension of groupoids fibered over N , the pull-back by f of $(\tilde{H} \rightarrow N)$ is isomorphic to $(\tilde{H}' \rightarrow N')$. The conclusion thus follows.

The last assertion is easy to check. \square

Lemma 4.9 *Let $c, c' \in \check{Z}^2(\sigma\mathcal{U}; \mathcal{S}^1)$, where \mathcal{U} is a cover of $(N \rtimes \Gamma)_\bullet$. Then $A_c \otimes_{C_0(\Gamma_0)} A_{c'} \xrightarrow{\sim} A_{c+c'}$ canonically.*

PROOF. If \mathcal{E} (resp. \mathcal{E}') is the S^1 -central extension whose C^* -algebra is A_c (resp. $A_{c'}$), then $A_{c+c'}$ is the C^* -algebra of $\mathcal{E} + \mathcal{E}'$. The latter is canonically isomorphic to $A_c \otimes_{C_0(\Gamma_0)} A_{c'}$. \square

Proposition 4.10 *Let M and N be Γ -spaces, c and c' \mathcal{S}^1 -valued Čech 2-cocycles on $M \rtimes \Gamma$ and $N \rtimes \Gamma$, respectively. Let $P = M \times_{\Gamma_0} N$ and $p_1 : P \rightarrow M$, $p_2 : P \rightarrow N$ be natural projections. Then $A_{p_1^*c + p_2^*c} \xrightarrow{\sim} A_c \otimes_{C_0(\Gamma_0)} A_{c'}$ canonically. Moreover this isomorphism is compatible with the maps Φ_b .*

PROOF. We have

$$\begin{aligned} A_{p_1^*c + p_2^*c'} &\cong A_{p_1^*c} \otimes_{C_0(P)} A_{p_2^*c'} \quad (\text{by Lemma 4.9 for } P \rtimes \Gamma) \\ &\cong p_1^*A_c \otimes_{C_0(P)} p_2^*A_{c'} \quad (\text{by Proposition 4.8}) \\ &\cong A_c \otimes_{C_0(\Gamma_0)} A_{c'}. \end{aligned}$$

\square

4.4 Equivariant twisted K -theory and the map Φ_b

First we recall the definition of twisted K -theory of a stack [43]. Let Γ be a locally compact groupoid with Haar system, \mathcal{U} an open cover of Γ_\bullet and $c \in \check{Z}^2(\sigma\mathcal{U}; \mathcal{S}^1)$ a 2-cocycle. Recall from Section 4.2 that c determines a C^* -algebra $C_r^*(\Gamma, c)$.

Definition 4.11 [43] The twisted K -theory group $K_c^i(\Gamma)$ is defined as $K^i(C_r^*(\Gamma, c))$.

Remark 4.12 In [43], we defined, for any $\alpha \in H^2(\Gamma_\bullet; \mathcal{S}^1)$, the twisted K -theory group $K_\alpha^i(\Gamma)$ as the K -theory group of the C^* -algebra of any S^1 -central extension representing α . This is indeed well-defined, as the C^* -algebras of two such central extensions are Morita equivalent. However, the shortcoming of such a definition is that the Morita equivalence is not canonical (see Proposition 4.15(3)). On the other hand, Definition 4.11 is canonical, as the 2-cocycle c determines a unique C^* -algebra rather than a Morita equivalence class of C^* -algebras. This is extremely important for our purpose in this paper since we need to study the functorial properties of twisted K -theory.

In particular, if $\Gamma \rightrightarrows \Gamma_0$ is a Lie groupoid acting on a manifold N with momentum map $J : N \rightarrow \Gamma_0$, and $c \in \check{Z}^2(\sigma\mathcal{U}; \mathcal{S}^1)$ is a 2-cocycle of the corresponding transformation groupoid $N \rtimes \Gamma : N \rtimes \Gamma \rightrightarrows N$. Then the twisted equivariant K -theory group is defined as follows [43]:

$$K_{c,\Gamma}^i(N) = K_c^i(N \rtimes \Gamma).$$

As an immediate consequence of Lemma 4.1, we have the following

Proposition 4.13 *For any open covers \mathcal{U} of Γ_\bullet , $c^1, c^2 \in \check{Z}^2(\sigma\mathcal{U}; \mathcal{S}^1)$, $b \in \check{C}^1(\sigma\mathcal{U}; \mathcal{S}^1)$ satisfying $c^1 - c^2 = \partial b$, there is a canonical isomorphism*

$$\Phi_{c^2,b,c^1}^{\mathcal{U}} : K_{c^1}^*(\Gamma) \rightarrow K_{c^2}^*(\Gamma),$$

which satisfies the following properties:

1. if $c^1 - c^2 = \partial b$ and $c^2 - c^3 = \partial b'$, then

$$\Phi_{c^3,b',c^2}^{\mathcal{U}} \circ \Phi_{c^2,b,c^1}^{\mathcal{U}} = \Phi_{c^3,b+b',c^1}^{\mathcal{U}};$$

2. for any $a \in \check{C}^0(\mathcal{U}, \mathcal{S}^1)$, we have

$$\Phi_{c,\partial a,c}^{\mathcal{U}} = \text{Id};$$

3. if \mathcal{U}' is a refinement of \mathcal{U} and c'^1, c'^2 and b' are the pull-backs of c^1, c^2 and b under the canonical map $\check{C}^*(\sigma\mathcal{U}'; \mathcal{S}^1) \rightarrow \check{C}^*(\sigma\mathcal{U}; \mathcal{S}^1)$, then $K_{c'_j}^*(\Gamma)$ is canonically isomorphic to $K_{c_j}^*(\Gamma)$, and the following diagram

$$\begin{array}{ccc} K_{c'^1}^*(\Gamma) & \longrightarrow & K_{c^1}^*(\Gamma) \\ \Phi_{c'^2,b',c'^1}^{\mathcal{U}'} \downarrow & & \downarrow \Phi_{c^2,b,c^1}^{\mathcal{U}} \\ K_{c'^2}^*(\Gamma) & \longrightarrow & K_{c^2}^*(\Gamma) \end{array}$$

commutes.

PROOF. (1) and (3) are immediate consequences of Lemma 4.1. For (2), on the groupoid level, according to Lemma 4.1,

$$\Phi_{c,\partial a,c}^{\mathcal{U}}(\gamma) = \gamma a(s(\gamma)) a(t(\gamma))^{-1} \quad , \quad \forall \gamma \in \tilde{\Gamma}^c.$$

This induces an automorphism of $C_r^*(\Gamma, c)$ given by $\Phi(f)(\gamma) = a(t(\gamma))^{-1} f(\gamma) a(s(\gamma))$, $\forall f \in C_c^\infty(\tilde{\Gamma}^c, L^c)$. Let $(Uf)(\gamma) = a(t(\gamma)) f(\gamma)$. Then U is an unitary multiplier of $C_r^*(\Gamma, c)$ and $\Phi(f) = U^* f U$. The conclusion thus follows. \square

Recall that $\check{C}^n(\Gamma_\bullet; \mathcal{S}^1) = \lim_{\mathcal{U}} \check{C}^n(\sigma\mathcal{U}; \mathcal{S}^1)$, where \mathcal{U} runs over covers of the form $(U_\gamma^p)_{\gamma \in \Gamma_p}$, $\gamma \in U_\gamma^p$, and \mathcal{U}' is finer than \mathcal{U} if $U_\gamma^p \subset U_\gamma^p$ for all p and all $\gamma \in \Gamma_p$. We are now ready to introduce

Definition 4.14 For any $c \in \check{Z}^2(\Gamma_\bullet; \mathcal{S}^1)$, we define the twisted K -theory group $K_c^i(\Gamma)$ as $K_{c'}^i(\Gamma)$, where $c' \in \check{Z}^2(\sigma\mathcal{U}'; \mathcal{S}^1)$ is a 2-cocycle on a cover \mathcal{U}' of the above form which corresponds to c in $\check{C}^2(\Gamma_\bullet; \mathcal{S}^1) = \lim_{\mathcal{U}} \check{C}^2(\sigma\mathcal{U}; \mathcal{S}^1)$.

Note that this definition is valid since if \mathcal{U}'' is another such cover, then $K_{c'}^i(\Gamma)$ and $K_{c''}^i(\Gamma)$ are canonically isomorphic according to Proposition 4.13(3). Now the following is an immediate consequence of Proposition 4.13.

Proposition 4.15 *Assume that c^1 and c^2 are two \mathcal{S}^1 -valued Čech 2-cocycles on a groupoid $\Gamma \rightrightarrows \Gamma_0$, and b is an \mathcal{S}^1 -valued 2-cochain such that $c^1 - c^2 = \partial b$. Then there is a canonical isomorphism*

$$\Phi_{c^2, b, c^1} : K_{c^1}^*(\Gamma) \rightarrow K_{c^2}^*(\Gamma),$$

which satisfies the following properties:

1. if $c^1 - c^2 = \partial b$ and $c^2 - c^3 = \partial b'$, then

$$\Phi_{c^3, b', c^2} \circ \Phi_{c^2, b, c^1} = \Phi_{c^3, b+b', c^1};$$

2. for any $a \in \check{C}^0(\Gamma_\bullet, \mathcal{S}^1)$, we have

$$\Phi_{c, b, c} = \Phi_{c, b + \partial a, c}.$$

As a consequence, we have

Corollary 4.16 *The map $[b] \mapsto \Phi_{c, b, c}$ defines a morphism $H^1(\Gamma; \mathcal{S}^1) \rightarrow \text{Aut}(K_c^*(\Gamma))$.*

For simplicity, in the sequel we will write Φ_b instead of Φ_{c^2, b, c^1} whenever there is no ambiguity.

4.5 External Kasparov product

Let us first recall a few basic facts about equivariant KK -theory, as introduced by Kasparov [33] and generalized by Le Gall to groupoids [36].

Assume that Γ is a locally compact σ -compact groupoid with Haar system (for instance, a Lie groupoid), A , A_1 , B , B_1 , and D are Γ - C^* -algebras, i.e C^* -algebras endowed with an action of Γ . Then there is a bifunctor

$$(A, B) \mapsto KK_\Gamma^i(A, B), \quad i = 0, 1$$

covariant in B and contravariant in A . There is also a suspension map

$$\sigma_{\Gamma_0, D} : KK_\Gamma^i(A, B) \rightarrow KK_\Gamma^i(D \otimes_{C_0(\Gamma_0)} A, D \otimes_{C_0(\Gamma_0)} B),$$

and an associative product

$$KK_\Gamma^i(A, B) \otimes KK_\Gamma^j(B, C) \rightarrow KK_\Gamma^{i+j}(A, C), \quad (53)$$

$$(\alpha, \beta) \mapsto \alpha \otimes_B \beta. \quad (54)$$

More generally, the map

$$KK_\Gamma^i(A, B \otimes_{C_0(\Gamma_0)} D) \otimes KK_\Gamma^j(A_1 \otimes_{C_0(\Gamma_0)} D, B_1) \rightarrow KK_\Gamma^{i+j}(A \otimes_{C_0(\Gamma_0)} A_1, B \otimes_{C_0(\Gamma_0)} B_1)$$

defined by

$$\alpha \otimes_D \beta := \sigma_{\Gamma_0, A_1}(\alpha) \otimes_{A_1 \otimes_{C_0(\Gamma_0)} B \otimes_{C_0(\Gamma_0)} D} \sigma_{\Gamma_0, B}(\beta)$$

is associative. In particular, when $D = C_0(\Gamma_0)$, we obtain an associative and (graded) commutative product

$$KK_\Gamma^i(A, B) \otimes KK_\Gamma^j(A_1, B_1) \rightarrow KK_\Gamma^{i+j}(A \otimes_{C_0(\Gamma_0)} A_1, B \otimes_{C_0(\Gamma_0)} B_1), \quad (55)$$

which is called the *external Kasparov product*.

We also recall [4, 41] that if A is a $\Gamma - C^*$ -algebra, there is a group morphism, the Baum–Connes assembly map

$$\mu_r : K_i^{top}(\Gamma; A) \rightarrow K_i(A \rtimes_r \Gamma).$$

Here the left-hand side is defined by $\lim_{Y \subset \underline{E}\Gamma} KK_\Gamma^i(C_0(Y), A)$, where Y runs over saturated subsets of a certain proper Γ -space $\underline{E}\Gamma$ such that Y/Γ is compact (if Γ is proper, then $\underline{E}\Gamma = \Gamma_0$). The map μ_r is defined by $\mu_r(x) = \lambda_{Y \rtimes \Gamma} \otimes_{C_0(Y) \rtimes_r \Gamma} j_{\Gamma, r}(x)$ for all $x \in KK_\Gamma^i(C_0(Y), A)$, where

$$j_{\Gamma, r} : KK_\Gamma^i(C_0(Y), A) \rightarrow KK^i(C_0(Y) \rtimes_r \Gamma, A \rtimes_r \Gamma)$$

is the Kasparov’s descent morphism and $\lambda_{Y \rtimes \Gamma} \in KK(\mathbb{C}, C_0(Y) \rtimes_r \Gamma) = K_0(C_r^*(Y \rtimes \Gamma))$ is a canonical element.

The Baum–Connes assembly map is an isomorphism when Γ is proper, or more generally, amenable [40] (as well as in many other cases such as connected Lie groups [17]). One does not however expect μ_r to be an isomorphism for every Hausdorff Lie groupoid, in view of the counterexamples in [30] (these counter examples are, however, neither Hausdorff nor smooth).

Proposition 4.17 *Assume that $\Gamma \rightrightarrows \Gamma_0$ is a proper Lie groupoid (or, more generally, a groupoid such that the Baum–Connes map is an isomorphism) acting on a space N . Let $c \in \check{Z}^2(\mathcal{U}, \mathcal{S}^1)$ be any 2-cocycle, where \mathcal{U} is a cover of $(N \rtimes \Gamma)_\bullet$ and $N \rtimes \Gamma$ is the transformation groupoid. Then there is a canonical isomorphism*

$$K_i^{top}(\Gamma; A_c) \xrightarrow{\sim} K_{c, \Gamma}^i(N), \quad (56)$$

where A_c is the C^* -algebra defined in Corollary 4.6. Moreover, under the same assumptions as in Proposition 4.15, denote again by α_{c^2, b, c^1} the element in $KK_\Gamma(A_{c^1}, A_{c^2})$ determined by the Γ -equivariant Morita equivalence $\alpha_{c^2, b, c^1} : A_{c^1} \rightarrow A_{c^2}$. Then we have a commutative diagram

$$\begin{array}{ccc} K_i^{top}(\Gamma; A_{c^1}) & \xrightarrow{\cong} & K_{c^1, \Gamma}^i(N) \\ \downarrow & & \downarrow \Phi_{c^2, b, c^1} \\ K_i^{top}(\Gamma; A_{c^2}) & \xrightarrow{\cong} & K_{c^2, \Gamma}^i(N), \end{array}$$

where the left vertical arrow is the multiplication by the element α_{c^2, b, c^1} .

PROOF. We have

$$K_i^{top}(\Gamma, A_c) \cong K_i(A_c \rtimes_r \Gamma) \cong K_i(C_r^*(N \rtimes \Gamma, c)) = K_{c, \Gamma}^i(N),$$

where the first isomorphism is the Baum–Connes assembly map μ_r , and the second one follows from Corollary 4.6. Note that each isomorphism above is canonical. That the diagram commutes follows from the equality $\mu_r(x \otimes \alpha_{c^2, b, c^1}) = j_{\Gamma, r}(\lambda_{Y \rtimes \Gamma} \otimes x \otimes \alpha_{c^2, b, c^1}) = j_{\Gamma, r}(\lambda_{Y \rtimes \Gamma} \otimes x) \otimes j_{\Gamma, r}(\alpha_{c^2, b, c^1})$ together with Corollary 4.6. \square

If $c \in \check{Z}^2(\Gamma_\bullet; \mathcal{S}^1)$ lives on a cover \mathcal{U} , and $c' \in \check{Z}^2(\sigma\mathcal{U}; \mathcal{S}^1)$ is the corresponding 2-cocycle, then $K_{c,\Gamma}^i(N) = K_i(A_{c'} \rtimes \Gamma)$ is canonically defined. Below, we will write A_c instead of $A_{c'}$ whenever there is no ambiguity (keeping in mind that the Γ - C^* -algebra A_c is determined up to a unique Morita equivalence).

Suppose now that Γ is proper. Let $N_2 = N \times_{\Gamma_0} N$, and $p_i : N_2 \rightarrow N$, $i = 1, 2$ be the natural projections. Take $A = C = C_0(Y)$, and $B = D = A_c$. Thus we have $B \otimes_{C_0(\Gamma_0)} D \cong A_{p_1^*c + p_2^*c}$ according to Proposition 4.10. Thus Proposition 4.17 together with the external Kasparov product Eq. (55) implies the following

Proposition 4.18 *Assume that $\Gamma \rightrightarrows \Gamma_0$ is a proper Lie groupoid acting on a space N . Let $c \in \check{Z}^2(\mathcal{U}, \mathcal{S}^1)$ be any 2-cocycle, where \mathcal{U} is a cover of $(N \rtimes \Gamma)_\bullet$ and $N \rtimes \Gamma$ is the transformation groupoid. We have a map*

$$K_{c,\Gamma}^i(N) \otimes K_{c,\Gamma}^j(N) \rightarrow K_{p_1^*c + p_2^*c, \Gamma}^{i+j}(N_2). \quad (57)$$

4.6 Gysin maps

Finally, we will need Gysin (wrong-way functoriality) maps. Such a map was studied extensively in [18, 31]. Recall that for manifolds M and N , a smooth map $f : M \rightarrow N$ is said to be K -oriented if the normal bundle $N_f = T^*M \oplus f^*(TN) \rightarrow M$ is K -oriented, i.e. it admits a Spin^c -structure. When f is a submersion, N_f can be replaced by $\ker(df) \subseteq TM$ [18].

Now let Γ be a Lie groupoid, and both M and N be Γ -manifolds. It is still not clear what will be an appropriate notion of Γ -equivariantly K -orientability for a general Γ -equivariant smooth map $f : M \rightarrow N$, since the normal bundle of f does not necessarily admit a Γ -action in general. However the Γ -equivariant K -orientability does make sense in any of the following special cases:

- (a) Γ possesses a pseudo-étale structure in the sense of Tang [39] (which is also called a flat structure by Behrend [5]). In particular, this includes the case that Γ is étale, or is a transformation groupoid corresponding to a Lie group action;
- (b) f is a submersion (in this case, $\ker(df)$ is endowed with an action of Γ);
- (c) both momentum maps $J : M \rightarrow \Gamma_0$ and $J' : N \rightarrow \Gamma_0$ are submersions. Denote by $VM = \ker J_* : TM \rightarrow T\Gamma_0$ and $VN = \ker J'_* : TN \rightarrow T\Gamma_0$ the vertical tangent bundles. Then f is said to be K -oriented if the vector bundle $V^*M \oplus f^*VN \rightarrow M$ admits a Γ -equivariant K -orientation.

From now on, we will assume that $f : M \rightarrow N$ is a Γ -equivariantly K -oriented submersion as in (b). Indeed, although Gysin maps in twisted K -theory may be constructed in other cases as well (see for instance [15]), we will only develop the case of submersions since other cases are not needed in this paper.

As we will see, f induces an element $f_! \in KK_{\Gamma}^d(C_0(M), C_0(N))$, where $d = \dim M - \dim N$, such that the relation $(g \circ f)_! = g_! \circ f_!$ holds. More generally, we have

Lemma 4.19 *Let Γ be a Lie groupoid, and M, N two proper Γ -manifolds. Assume that $f : M \rightarrow N$ is a Γ -equivariantly K -oriented submersion. Then for any \mathcal{S}^1 -valued Čech*

2-cocycle c on the groupoid $N \rtimes \Gamma$, there is a Gysin element $f_!^c \in KK_\Gamma^d(A_{f^*c}, A_c)$, where $d = \dim N - \dim M$, satisfying the property:

$$f_!^{g^*c} \otimes_{A_{g^*c}} g_!^c = (g \circ f)_!^c, \quad (58)$$

for any Γ -equivariant K -oriented maps $f : M \rightarrow N$ and $g : N \rightarrow P$. The K -orientation of $g \circ f$ is induced from that of f and g . Here both sides are considered as elements in $KK_\Gamma^{d''}(A_{(g \circ f)^*c}, A_c)$ and $d'' = \dim P - \dim M$.

PROOF. It is standard that any K -oriented submersion $f : M \rightarrow N$ yields a Gysin element $f_! \in KK^d(C_0(M), C_0(N))$ [18, 31]. When Γ is a Lie group, an equivariant version was proved by Kasparov-Skandalis [34, §4.3]: any Γ -equivariantly K -oriented map $f : M \rightarrow N$ determines an element $f_! \in KK_\Gamma^d(C_0(M), C_0(N))$. A similar argument can be adapted to show that the same assertion holds when Γ is a Lie groupoid, and KK_Γ is Le Gall's groupoid equivariant KK -theory [36]. In fact, since f is also $N \rtimes \Gamma$ -equivariant, it is easy to see that one obtains an element $f_! \in KK_{N \rtimes \Gamma}^d(C_0(M), C_0(N))$.

Consider

$$\sigma_{N, A_c}(f_!) \in KK_{N \rtimes \Gamma}^d(C_0(M) \otimes_{C_0(N)} A_c, C_0(N) \otimes_{C_0(N)} A_c).$$

By Proposition 4.8, we have $C_0(M) \otimes_{C_0(N)} A_c \cong A_{f^*c}$. Thus $\sigma_{N, A_c}(f_!) \in KK_{N \rtimes \Gamma}^d(A_{f^*c}, A_c)$. We define $f_!^c$ as the image of this element under the forgetful functor $KK_{N \rtimes \Gamma} \rightarrow KK_\Gamma$.

We need to check that $f_!^c$ is well-defined. Recall that A_c is defined only up to Morita equivalence. In the definition of A_c , we have implicitly chosen an open cover \mathcal{U} , on which the 2-cocycle c lives. Assume that c' is another such a 2-cocycle, which is defined on another cover \mathcal{U}' . Let $\alpha \in KK_{N \rtimes \Gamma}(A_c, A_{c'})$ be the element determined by the Morita equivalence between A_c and $A_{c'}$. Since all the constructions are natural, $f^*\alpha \in KK_{M \rtimes \Gamma}(A_{f^*c}, A_{f^*c'}) = KK_{M \rtimes \Gamma}(C_0(M) \otimes_{C_0(N)} A_c, C_0(M) \otimes_{C_0(N)} A_{c'})$ is the element induced by the Morita equivalence between A_{f^*c} and $A_{f^*(c')}$.

We need to show that $f_!^c$ and $f_!^{c'}$ can be identified, i.e.

$$\sigma_{N, A_c}(f_!) \otimes_{A_c} \alpha = f^*\alpha \otimes_{f^*A_{c'}} \sigma_{N, A_{c'}}(f_!) \in KK_{N \rtimes \Gamma}^*(f^*A_c, A_{c'}).$$

The left hand side is $f_! \otimes_{C_0(N)} \alpha$, while the right hand side is equal to $\sigma_{N, C_0(M)}(\alpha) \otimes_{C_0(M) \otimes_{C_0(N)} A_{c'}} \sigma_{N, A_{c'}}(f_!) = \alpha \otimes_{C_0(N)} f_!$. Thus the equality follows from the commutativity of the external Kasparov product.

Eq. (58) now follows from the compatibility of the suspension σ with the Kasparov product

$$f_!^{g^*c} \otimes_{A_{g^*c}} g_!^c = \sigma_{P, A_c}(f_!) \otimes_{A_{g^*c}} \sigma_{P, A_c}(g_!) = \sigma_{P, A_c}(f_! \otimes_{C_0(N)} g_!) = \sigma_{P, A_c}((g \circ f)_!) = (g \circ f)_!^c.$$

□

Corollary 4.20 *Under the same hypothesis as in Lemma 4.19, there is a Gysin map*

$$f_! : K_{f^*c, \Gamma}^i(M) \rightarrow K_{c, \Gamma}^{i+d}(N),$$

with $d = \dim N - \dim M$, which satisfies $g_! \circ f_! = (g \circ f)_!$.

PROOF. We have $f_!(\beta) = \beta \otimes_{f^*A_c} f_!^c$. □

The following proposition describes the naturality property of the Gysin map with respect to the cocycle c .

Proposition 4.21 *Under the same hypothesis as in Corollary 4.20, assume that c' is another 2-cocycle which is cohomologous to c , i.e. $c - c' = \partial u$ for some $u \in \check{C}^1((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$. Then the diagram*

$$\begin{array}{ccc} K_{f^*c, \Gamma}^i(M) & \xrightarrow{f_!} & K_{c, \Gamma}^{i+d}(N) \\ \downarrow \Phi_{f^*u} & & \downarrow \Phi_u \\ K_{f^*c', \Gamma}^i(M) & \xrightarrow{f_!} & K_{c', \Gamma}^{i+d}(N) \end{array}$$

commutes.

PROOF. We can assume that the relation $c - c' = \partial u$ holds in some complex $C^*(\sigma\mathcal{U}; \mathcal{S}^1)$. Therefore the isomorphism Φ_u is implemented by an isomorphism of S^1 -central extensions, and thus by an invertible element $\alpha \in KK_{N \rtimes \Gamma}(A_c, A_{c'})$. It is clear, by naturality of the constructions, that Φ_{f^*u} is implemented by $f^*\alpha \in KK_{M \rtimes \Gamma}(A_{f^*c}, A_{f^*c'})$. Then for all $\beta \in K_{f^*c, \Gamma}^*(M)$, we have $(\Phi_u \circ f_!)(\beta) = \beta \otimes_{f^*A_c} f_!^c \otimes_{A_c} \alpha$, while $(f_! \circ \Phi_{f^*u})(\beta) = \beta \otimes_{f^*A_c} f^*\alpha \otimes_{f^*A_{c'}} f_!^{c'}$. Thus we need the identity $\sigma_{N, A_c}(f_!) \otimes_{A_c} \alpha = f^*\alpha \otimes_{f^*A_{c'}} \sigma_{N, A_{c'}}(f_!) \in KK_{N \rtimes \Gamma}^*(f^*A_c, A_{c'})$, which can be proved exactly in the same way as in Proposition 4.19. \square

4.7 Ring structure on equivariant twisted K -theory group

Let $N \xrightarrow{\varphi} \Gamma$ be a crossed module, where $\Gamma \rightrightarrows \Gamma_0$ is a proper Lie groupoid such that $s : N \rightarrow N_0$ is Γ -equivariantly K -oriented. Assume that (c, b, a) is a multiplier as in Definition 3.15.

Definition 4.22 Define

$$K_{c, \Gamma}^{i+d}(N) \otimes K_{c, \Gamma}^{j+d}(N) \rightarrow K_{c, \Gamma}^{i+j+d}(N),$$

where $d = \dim N - \dim N_0$, as the composition of the external Kasparov product $K_{c, \Gamma}^{i+d}(N) \otimes K_{c, \Gamma}^{j+d}(N) \rightarrow K_{p_1^*c + p_2^*c, \Gamma}^{i+j}(N_2)$ as in Proposition 4.18 with the maps

$$K_{p_1^*c + p_2^*c, \Gamma}^{i+j}(N_2) \xrightarrow{\Phi_b} K_{m^*c, \Gamma}^{i+j}(N_2) \xrightarrow{m_!} K_{c, \Gamma}^{i+j+d}(N).$$

Here $m_!$ is the Gysin map corresponding to $m : N_2 \rightarrow N$.

Note that in the above definition if c is defined on some cover \mathcal{U} , then p_1^*c and p_2^*c are defined on different covers. Therefore to define $p_1^*c + p_2^*c$, one needs to pass to a refinement. This is the reason why we choose to define twisted K -theory with respect to a Čech cocycle $c \in \check{C}^2((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ as in Definition 4.14 instead of a cocycle in $\check{C}^2(\sigma\mathcal{U}; \mathcal{S}^1)$ for some fixed cover.

Also, note that $s : N \rightarrow N_0$ is Γ -equivariantly K -oriented implies that $pr_2 : N \times_{N_0} N \rightarrow N$ is Γ -equivariantly K -oriented. Since N_2 is Γ -equivariantly diffeomorphic to $N \times_{N_0} N$ by $\psi : (x, y) \mapsto (x, xy)$, it follows that $m = pr_2 \circ \psi : N_2 \rightarrow N$ is Γ -equivariantly K -oriented. Moreover, define maps $N_3 \rightarrow N_2$ by $m_{12}(x, y, z) = (xy, z)$ and $m_{23}(x, y, z) = (x, yz)$. It is not hard to check from the assumptions that the K -orientations of $m \circ m_{12}$ and $m \circ m_{23}$ coincide. Let $m_{123} = m \circ m_{12} = m \circ m_{23}$ be endowed with this K -orientation, then we obtain the following

Proposition 4.23

$$m_{12!} \circ m_! = m_{123!} = m_{23!} \circ m_!.$$

Remark 4.24 In Definition 4.22 above, if N is not connected, $K_{c,\Gamma}^{i+d}(N)$ needs to be replaced by $\oplus_l K_{(c,\Gamma)}^{i+d_l}(N^{[l]})$, where $N = \coprod_l N^{[l]}$ is the decomposition into its connected components, and d_l is the dimension of $N^{[l]}$. An example where a disconnected crossed module naturally appears is when N is the space of closed loops $S\Gamma = \{g \in \Gamma \mid s(g) = t(g)\}$, and Γ is an étale proper groupoid. The results formulated below are still valid in this context.

Now we can state the main theorem of this paper.

Theorem 4.25 *Let $N \xrightarrow{\varphi} \Gamma$ be a crossed module, where $\Gamma \rightrightarrows \Gamma_0$ is a proper Lie groupoid such that $s : N \rightarrow N_0$ is Γ -equivariantly K -oriented. Assume that (c, b, a) is a multiplicative triple, where $c \in \check{C}^2((N \rtimes \Gamma)_\bullet, \mathcal{S}^1)$, $b \in \check{C}^1((N_2 \rtimes \Gamma)_\bullet, \mathcal{S}^1)$ and $a \in \check{C}^0((N_3 \rtimes \Gamma)_\bullet, \mathcal{S}^1)$. Then the product*

$$K_{c,\Gamma}^{i+d}(N) \otimes K_{c,\Gamma}^{j+d}(N) \rightarrow K_{c,\Gamma}^{i+j+d}(N)$$

is associative.

PROOF. Let us study the product $(x, y, z) \mapsto x(yz)$ from $K_{c,\Gamma}^{i+d}(N) \otimes K_{c,\Gamma}^{j+d}(N) \otimes K_{c,\Gamma}^{k+d}(N)$ to $K_{c,\Gamma}^{i+j+k+d}(N)$. To simplify notations, we assume $i = j = k = 0$. The product $x(yz)$ is obtained by composing the external product $K_{c,\Gamma}^d(N) \otimes K_{c,\Gamma}^d(N) \otimes K_{c,\Gamma}^d(N) \rightarrow K_{p_1^*c+p_2^*c+p_3^*c,\Gamma}^d(N_3)$ with $\Phi_{p_{23}^*b}$, followed by the down-right composition of the diagram below:

$$\begin{array}{ccccc} K_{p_1^*c+m_{23}^*c,\Gamma}^d(N_3) & \xrightarrow{\Phi_{m_{23}^*b}} & K_{m_{123}^*c,\Gamma}^d(N_3) & & \\ (m_{23})! \downarrow & & \downarrow (m_{23})! & \searrow (m_{123})! & \\ K_{p_1^*c+p_2^*c,\Gamma}^0(N_2) & \xrightarrow{\Phi_b} & K_{m^*c,\Gamma}^0(N_2) & \xrightarrow{m_!} & K_{c,\Gamma}^d(N). \end{array}$$

(Recall that $m_{23}(x, y, z) = (x, yz)$, $m_{12}(x, y, z) = (xy, z)$, $m_{123}(x, y, z) = xyz$.)

It follows from Proposition 4.21, Corollary 4.20 and Proposition 4.23 that the diagram above commutes. Hence the product $x(yz)$ can also be obtained from the composition of the external product $K_{c,\Gamma}^d(N) \otimes K_{c,\Gamma}^d(N) \otimes K_{c,\Gamma}^d(N) \rightarrow K_{p_1^*c+p_2^*c+p_3^*c,\Gamma}^d(N_3)$ with the maps

$$K_{p_1^*c+p_2^*c+p_3^*c,\Gamma}^d(N_3) \xrightarrow{\Phi_{p_{23}^*b+m_{23}^*b}} K_{m_{123}^*c,\Gamma}^d(N_3) \xrightarrow{(m_{123})!} K_{c,\Gamma}^d(N).$$

Similarly, the product $(xy)z$ is obtained by composing the external product $K_{c,\Gamma}^d(N) \otimes K_{c,\Gamma}^d(N) \otimes K_{c,\Gamma}^d(N) \rightarrow K_{p_1^*c+p_2^*c+p_3^*c,\Gamma}^d(N_3)$ with the maps

$$K_{p_1^*c+p_2^*c+p_3^*c,\Gamma}^d(N_3) \xrightarrow{\Phi_{p_{12}^*b+m_{12}^*b}} K_{m_{123}^*c,\Gamma}^d(N_3) \xrightarrow{(m_{123})!} K_{c,\Gamma}^d(N).$$

Now since $p_{23}^*b + m_{23}^*b - p_{12}^*b + m_{12}^*b = \partial^*b = \partial a$, it follows from Proposition 4.15 that $\Phi_{p_{23}^*b+m_{23}^*b} = \Phi_{p_{12}^*b+m_{12}^*b}$. This completes the proof of the theorem. \square

4.8 Ring structure on the K -theory group twisted by 2-gerbes

Thanks to the transgression maps, for any crossed module $N \rightarrow \Gamma$, one can produce a canonical multiplier from $\check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$. Therefore one obtains a ring structure on the corresponding twisted K -theory group.

More precisely, given any $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$, $T_1 e \in \check{Z}^2((N \rtimes \Gamma)_\bullet; \mathcal{S}^1)$ is 2-multiplicative. Define

$$K_{e,\Gamma}^*(N) := K_{T_1 e, \Gamma}^*(N).$$

Let $c = T_1 e$, $b = -T_2 e$ and $a = -T_3 e$. Then (c, b, a) is a multiplier according to Corollary 3.14. Thus $K_{e,\Gamma}^*(N)$ naturally admits a ring structure.

Theorem 4.26 *Let $N \xrightarrow{\varphi} \Gamma$ be a crossed module, where $\Gamma \rightrightarrows \Gamma_0$ is a proper Lie groupoid such that $s : N \rightarrow N_0$ is Γ -equivariantly K -oriented. Let $d = \dim N - \dim N_0$ (see also Remark 4.24).*

1. *For any $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$, the twisted K -theory group $K_{e,\Gamma}^*(N)$ is endowed with a ring structure*

$$K_{e,\Gamma}^{i+d}(N) \otimes K_{e,\Gamma}^{j+d}(N) \rightarrow K_{e,\Gamma}^{i+j+d}(N),$$

where $d = \dim N - \dim N_0$.

2. *Assume that e and $e' \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$ satisfy $e - e' = \partial u$ for some $u \in \check{C}^2(\Gamma_\bullet; \mathcal{S}^1)$. Then there is a ring isomorphism*

$$\Psi_{e',u,e} : K_{e,\Gamma}^*(N) \rightarrow K_{e',\Gamma}^*(N)$$

such that

- if $e - e' = \partial u$ and $e' - e'' = \partial u'$, then

$$\Psi_{e'',u',e' \circ e} \Psi_{e',u,e} = \Psi_{e'',u+u',e};$$

- for any $v \in \check{C}^1(\Gamma_\bullet; \mathcal{S}^1)$,

$$\Psi_{e',u,e} = \Psi_{e',u+\partial v,e}.$$

3. *There is a morphism*

$$H^2(\Gamma_\bullet; \mathcal{S}^1) \rightarrow \text{Aut } K_{e,\Gamma}^*(N).$$

The ring structure on $K_{e,\Gamma}^{+d}(N)$, up to an isomorphism, depends only on the cohomology class $[e] \in H^3(\Gamma_\bullet; \mathcal{S}^1)$. The isomorphism is unique up to an automorphism of $K_{e,\Gamma}^{*+d}(N)$ induced from $H^2(\Gamma_\bullet; \mathcal{S}^1)$.*

PROOF. Let $c = T_1 e$ and $c' = T_1 e'$. Since ∂ and T_1 anti-commute according to Eq. (23), we have $c - c' = -\partial T_1 u$. Define

$$\Psi_{e',u,e} = \Phi_{c',-T_1 u,c}.$$

All the assertions of the theorem except that $\Psi_{e',u,e}$ is a ring morphism, follow from Proposition 4.15.

It remains to prove that $\Psi_{e',u,e}$ preserves the product. We thus need to check that the following diagram:

$$\begin{array}{ccccccc}
K_{c,\Gamma}^*(N) \otimes K_{c,\Gamma}^*(N) & \longrightarrow & K_{p_1^*c+p_2^*c,\Gamma}^*(N_2) & \xrightarrow{\Phi_b} & K_{m^*c,\Gamma}^*(N_2) & \xrightarrow{m!} & K_{c,\Gamma}^*(N) \\
\downarrow \Phi_{-T_1u} \otimes \Phi_{-T_1u} & & \downarrow \Phi_{-p_1^*T_1u-p_2^*T_1u} & & \downarrow \Phi_{-m^*T_1u} & & \downarrow \Phi_{-T_1u} \\
K_{c',\Gamma}^*(N) \otimes K_{c',\Gamma}^*(N) & \longrightarrow & K_{p_1^*c'+p_2^*c',\Gamma}^*(N_2) & \xrightarrow{\Phi_{b'}} & K_{m^*c',\Gamma}^*(N_2) & \xrightarrow{m!} & K_{c',\Gamma}^*(N)
\end{array}$$

commutes.

The commutativity of the third square follows from Proposition 4.21, and the commutativity of the first square follows from Proposition 4.10.

For the second square, it suffices, using Proposition 4.15, to check that $b - m^*T_1u$ and $-p_1^*T_1u - p_2^*T_1u + b'$ differ by a Čech coboundary. Now we have, using the relation $\partial'T_1 = T_2\partial - \partial T_2$ (see Eq. (15)) that,

$$\begin{aligned}
(b - m^*T_1u) - (-p_1^*T_1u - p_2^*T_1u + b') &= b - b' + \partial'T_1u \\
&= -T_2(e - e') + \partial'T_1u = -T_2\partial u + \partial'T_1u = -\partial T_2u.
\end{aligned}$$

This concludes the proof. \square

As an application, we consider twisted K -theory group of an inertia groupoid. Let $\Gamma \rightrightarrows \Gamma_0$ be a Lie groupoid and consider the crossed module $S\Gamma \rightarrow \Gamma$. As before, $\Lambda\Gamma : S\Gamma \rtimes \Gamma \rightrightarrows S\Gamma$ denote the inertia groupoid of Γ . Any element in the image of the transgression map $T_1 : H^3(\Gamma_\bullet; \mathcal{S}^1) \rightarrow H^2(\Lambda\Gamma_\bullet; \mathcal{S}^1)$ is 2-multiplicative according to Remark 3.13. Thus one obtains a ring structure on the corresponding twisted K -theory group. Since $H^3(\Gamma_\bullet; \mathcal{S}^1)$ classifies 2-gerbes [9], we conclude that the twisted K -theory group on the inertia stack twisted by a 2-gerbe over the stack admits a ring structure.

Theorem 4.27 *Let $\Gamma \rightrightarrows \Gamma_0$ be a proper Lie groupoid such that $S\Gamma$ is a manifold and $S\Gamma \rightarrow \Gamma_0$ is Γ -equivariantly K -oriented (these assumptions hold for instance when Γ is proper and étale, or when Γ is a compact connected and simply connected Lie group). Let $d = \dim S\Gamma - \dim \Gamma_0$ (see also Remark 4.24).*

1. *For any $e \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$, the twisted K -theory group $K_{e,\Gamma}^{*+d}(S\Gamma)$ is endowed with a ring structure*

$$K_{e,\Gamma}^{i+d}(S\Gamma) \otimes K_{e,\Gamma}^{j+d}(S\Gamma) \rightarrow K_{e,\Gamma}^{i+j+d}(S\Gamma).$$

2. *Assume that e and $e' \in \check{Z}^3(\Gamma_\bullet; \mathcal{S}^1)$ satisfy $e - e' = \partial u$ for some $u \in \check{C}^2(\Gamma_\bullet; \mathcal{S}^1)$. Then there is a ring isomorphism*

$$\Psi_{e',u,e} : K_{e,\Gamma}^*(S\Gamma) \rightarrow K_{e',\Gamma}^*(S\Gamma)$$

satisfying the properties:

- *if $e - e' = \partial u$ and $e' - e'' = \partial u'$, then*

$$\Psi_{e'',u',e'} \circ \Psi_{e',u,e} = \Psi_{e'',u+u',e}$$

- *for any $v \in \check{C}^1(\Gamma_\bullet; \mathcal{S}^1)$,*

$$\Psi_{e',u,e} = \Psi_{e',u+\partial v,e}.$$

3. There is a morphism

$$H^2(\Gamma_\bullet; \mathcal{S}^1) \rightarrow \text{Aut } K_{e,\Gamma}^*(ST).$$

The ring structure on $K_{e,\Gamma}^{*+d}(ST)$, up to an isomorphism, depends only on the cohomology class $[e] \in H^3(\Gamma_\bullet; \mathcal{S}^1)$. The isomorphism is unique up to an automorphism of $K_{e,\Gamma}^{*+d}(ST)$ induced from $H^2(\Gamma_\bullet; \mathcal{S}^1)$.

Remark 4.28 In [1], Adem-Ruan-Zhang introduced an associative stringy product for the twisted orbifold K -theory of a compact, almost complex orbifold \mathcal{X} . This product is defined on the twisted K -theory of the inertia orbifold $\Lambda\mathcal{X}$, where the twisting class τ is assumed to be in the image of the transgression $H^4(\mathcal{X}_\bullet, \mathbb{Z}) \rightarrow H^3(\Lambda\mathcal{X}_\bullet, \mathbb{Z})$. It would be interesting to investigate the relation between the Adem-Ruan-Zhang's stringy product and the product we introduced above.

Now assume that G is a compact, connected and simply connected simple Lie group. Then $SG \cong G$ and the G -action on G is the conjugation.

Lemma 4.29 G admits a G -equivariant Spin^c structure, and thus a G -equivariant Spin^c -structure.

PROOF. Identify TG with $G \times \mathfrak{g}$ by left translations, where \mathfrak{g} is the Lie algebra of G . Under this identification, the G -action on TG , which is the lift of the conjugation action on G , becomes $g \cdot (x, T) = (gxg^{-1}, \text{Ad}(g)T)$, $\forall g \in G$ and $(x, T) \in G \times \mathfrak{g}$. Endow \mathfrak{g} with an Ad -invariant inner product. Since G is connected and simply connected, the group morphism $\text{Ad} : G \rightarrow SO(\mathfrak{g})$ thus lifts to $G \rightarrow \widetilde{SO}(\mathfrak{g})$. \square

According to Corollary 3.20, the transgression map $T_1 : H^3(G_\bullet; \mathcal{S}^1) \rightarrow H^2((G \rtimes G)_\bullet; \mathcal{S}^1)$ is an isomorphism. Thus every cohomology class in $H^2((G \rtimes G)_\bullet; \mathcal{S}^1)$ can be represented by a 2-multiplicative cocycle. Moreover, it is known that $H^2(G_\bullet; \mathcal{S}^1) = 0$. Thus we are led to the following

Theorem 4.30 Let G be a compact, connected and simply connected simple Lie group, and $[c] \in H^2((G \rtimes G)_\bullet; \mathcal{S}^1) \cong \mathbb{Z}$. Then the equivariant twisted K -theory group $K_{[c],G}^*(G)$ is endowed with a canonical ring structure

$$K_{[c],G}^{i+d}(G) \otimes K_{[c],G}^{j+d}(G) \rightarrow K_{[c],G}^{i+j+d}(G),$$

where $d = \dim G$, in the sense that there is a canonical isomorphism of the rings when using any two 2-cocycles in $\check{Z}^2((G \rtimes G)_\bullet; \mathcal{S}^1)$ which are images under the transgression T_1 .

Remark 4.31 In general, if G is a compact and connected Lie group, G admits a G -equivariant Spin^c -structure if and only if there exists an infinitesimal character $\psi : \mathfrak{g} \rightarrow \text{Lie}(U(1))$ such that $\rho_G + \psi$ is a weight of T , where ρ_G is the half-sum of positive roots (for any choice of maximal torus T)⁶. In that case, $e \in \check{Z}^3((G \rtimes G)_\bullet; \mathcal{S}^1)$ determines a ring structure on $K_{[T_1 e],G}^*(G)$, but this ring structure a priori depends on the choice of the cocycle e and not just of the cohomology class $[e] \in H^3((G \rtimes G)_\bullet; \mathcal{S}^1)$, since $H^2(G_\bullet; \mathcal{S}^1) \neq 0$ in general.

⁶We are grateful to Meinrenken for pointing this out to us.

Remark 4.32 1. There is a similar product in K -homology. Let G be a compact group, $N \rightarrow G$ a crossed module such that N is G -equivariantly K -oriented. Let us define $K_{i,c,G}(N)$ as $KK_G^i(A_c, \mathbb{C})$. One can define an associative product as the composition of the external product $K_{i,c,G}(N) \otimes K_{j,c,G}(N) \rightarrow K_{i+j,p_1^*c+p_2^*c,G}(N)$, the isomorphism $K_{i+j,p_1^*c+p_2^*c,G}(N) \rightarrow K_{i+j,m^*c,G}(N)$ given by b such that $\partial'c = \partial b$, and the map $m^* : K_{i+j,m^*c,G}(N) \rightarrow K_{i+j,c,G}(N)$ coming from $m : A_c \rightarrow A_{m^*c}$.

It should not be hard to prove that the product in K -theory and the one in K -homology are related by Poincaré duality.

2. For the case $M = G$ and $c = 0$, $K_G^*(G)$ is explicitly computed by Brylinski-Zhang [13]. It would be interesting to investigate how to express the ring structure $K_G^i(G) \otimes K_G^j(G) \rightarrow K_G^{i+j+d}(G)$ explicitly in this case.
3. Freed-Hopkins-Teleman have proved a remarkable theorem [26, 28] that the equivariant twisted K -theory group is isomorphic to Verlinde algebra for a compact connected Lie group. See also [16] for related discussion from a different perspective. Here we define equivariant twisted K -theory group from a different viewpoint by using K -theory of groupoid C^* -algebras. It would be interesting to explore the connection between the ring structure on $K_{[c],G}^*(G)$ using our construction and the ones in [26] and [16].

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